Heteroclinic primary intersections and codimension one Melnikov method for volume-preserving maps

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We study families of volume preserving diffeomorphisms in $\mathbb{R}^3$ that have a pair of hyperbolic fixed points with intersecting codimension one stable and unstable manifolds. Our goal is to elucidate the topology of the intersections and how it changes with the parameters of the system. We show that the ‘‘primary intersection’’ of the stable and unstable manifolds is generically a neat submanifold of a ‘‘fundamental domain.’’ We compute the intersections perturbatively using a codimension one Melnikov function. Numerical experiments show various bifurcations in the homotopy class of the primary intersections. © *2000 American Institute of Physics.* [S1054-1500(00)01201-5]

The theory of transport for area-preserving maps is based on the construction of ‘‘partial barriers,’’ typically from segments of stable and unstable manifolds of fixed points, periodic or quasiperiodic orbits. Our ultimate goal is the generalization of this theory to higher dimensions. Perhaps the simplest place to start is with volume-preserving maps in three dimensions. A hyperbolic fixed point of such a map has either a two-dimensional stable or unstable manifold. Since they are codimension one, these manifolds can separate phase space into regions containing nontrivial invariant sets. The major problem is to choose appropriate domains of these manifolds that can be used in the construction of partial barriers. To this end we define fundamental domains and their primary intersections by using a partial ordering along the manifolds. Primary intersections are typically curves on the two-dimensional manifolds. These curves, when restricted to a fundamental domain, become loops and can be classified by their homotopy. As parameters of a map change, these homotopy classes can change as well. To investigate this, we start with an integrable mapstar304.8(4.8(has)well.)-4404.8asmethod to compute the splitting distance betw manifolds. Our numerical computations show the creation and destruction of intersection loops of various types.

I. INTRODUCTION

Volume-preserving maps provide an interesting and nontrivial class of dynamical systems that give perhaps the simplest, natural generalization of the class of area-preserving maps to higher dimensions. Moreover, volume-preserving maps naturally arise in applications as the time one Poincaré map of incompressible flows—even when the vector field of the flow is nonautonomous. Thus the study of the dynamics of volume-preserving maps is of interest dedef de57owuid-338.9(ofd)-2J exclude or include endpoints of these segments as appropriate). A point x is a *primary* tively. The saddle-node and period doubling lines divide the (t,s) plane into quadrants which alternate between type A and B. The dynamics on the two-dimensional manifolds will depend upon whether the pair of multipliers are complex or are real.

If a map has a pair of fixed points, one of type A and one of type B and the pair of two-dimensional manifolds (stable and unstable) intersect, then generically they intersect along one-dimensional manifolds. We have observed earlier[3] changes in the topology of the intersection manifolds as the parameters vary. Elucidating this topology is the primary aim of this paper.

III. PRIMARY INTERSECTIONS

In this section we introduce the concepts of the fundamental domain of a stable (or unstable) manifold and of primary heteroclinic intersections between such manifolds. These generalize the well known concepts for two dimensional maps. We, as usual, assume that $f:\mathbb{R}^3\to\mathbb{R}^3$ preserves the 3-form Ω, (1).

A. Proper loops and fundamental domains

Definition 1 (Proper Loop): *Suppose $a=f(a)$ is hyperbolic and of type A, i.e., has a two-dimensional stable manifold $W^s(a)$. A proper loop $\gamma\subset W^s(a)$ is a curve that bounds a local submanifold that is an isolating neighborhood of a. In other words γ is proper if there is an open local submanifold $W^s_{loc}(a)$ such that*

(a) $\partial W^s_{loc}(a)=\gamma$ and

(b) $f(\overline{W^s_{loc}(a)})\subset int(W^s_{loc}(a))$.

Similarly if b is a type B fixed point, then a loop $\subset W^u(b)$ is proper if it is proper for f^{-1}.

If γ is proper, we can define the stable manifold *starting* at γ, denoted by by $W^s_\gamma(a)$, as the closure in $W^s(a)$ of the local submanifold bounded by γ in Defn. 1. Similarly, if b is a type B fixed point with a proper loop , we define the manifold *up to* , denoted $W^u(b)$, as the interior of the local manifold that corresponds to f^{-1} in Defn. 1.

Notice that the definition is not symmetric, because $W^s_\gamma(a)$ is a closed subset of $W^s(a)$, while $W^u(b)$ is open in $W^u(b)$ (cf. Fig. 2). The asymmetry is just a technicality in order to simplify some proofs.

Definition 2 (Fundamental domain): *Let a and b be hyperbolic fixed points of type A and B, respectively. An annulus is a fundamental domain of $W^s(a)$ if there exists some proper loop γ in $W^s(a)$, such that*

$$ = W^s_{\gamma}(a)\setminus W^s_{f\circ\gamma}(a).$$

Similarly, a fundamental domain in $W^u(b)$ is a manifold with boundary of the form

$$ = W^u(b)\setminus W^u_{f^{-1}\circ}(b),$$

where is a proper loop in $W^u(b)$. In addition, we define $\mathcal{F}^u(b)$ as the set of all fundamental domains in $W^u(b)$, and $\mathcal{F}^s(a)$ as the set of all fundamental domains in $W^s(a)$.

In each case, the fundamental domain is an annulus with one open and one closed edge. An immediate consequence of the definition is that all the forward and backward iterations of a fundamental domain are also fundamental. It is easy to see that proper loops always exist, and in fact, the stable (and unstable) manifolds can be decomposed as the disjoint union of fundamental domains:

$$W^s(a)=\bigcup_{k\in Z} f^k(\ _\gamma(a)).$$

The importance of fundamental domains is that much of the information about the entire manifold can be found by looking only at these annuli. For instance, the *primary heteroclinic intersection* between $W^s(a)$ and $W^u(b)$, which we define next, is defined using fundamental domains.

B. Primary intersection

Given a fundamental domain , the points $\xi\in W^s(a)$ are given a partial order defined by the integer k

Lemma 1: *Suppose* $W^s(a)\cap W^u(b)$ 0. *Then for all* $\in \mathcal{F}^s(a)$ *and* $\in \mathcal{F}^u(b)$, *there exists a unique integer* , *called the intersection index such that*

$$)\equiv \sup\{k\in \mathbb{Z}: \ \cap f^k \ ) \ \varnothing\}$$
$$=\sup\{k\in \mathbb{Z}: f^{-k} \ )\cap \ \varnothing\}.$$

Proof: This follows from the facts that each manifold is composed of the union of the fundamental domains, that the closures of and are compact and do not contain the fixed points, and that $f^k(\ )\to a$ and $f^{-k}(\ )\to b$ as $k\to\infty$. The two definitions are equivalent, since $f^{-k}(\ \cap f^k(\ ))$ $=f^{-k}(\ )\cap$. ■

The intersection index is useful because it is invariant: $(f(\ ),f(\ ))=\ (\ ,\ )$. More generally, the intersection index of iterates of fundamental domains changes as

$$f^m \ ),f^n \ ))= \ )+m-n.$$

Roughly speaking, a primary intersection is the set of points where the stable and unstable manifolds ''first'' meet. For maps of the plane, one says that $x\in W^s(a)\cap W^u(b)$ is a primary intersection point if the intersection of the stable manifold starting at x and the unstable manifold up to x is empty: $W^s_x(a)\cap W^u_x(b)=\varnothing$. This means that one can choose fundamental domains and so that that their boundaries are primary) heteroclinic points. As noted by Easton, this leads to a classification of heteroclinic orbits by their ''type,''[27] and subsequently a classification of the structure of the ''trellis,'' the closure of the stable and unstable manifolds.

To directly generalize the planar definition, we would need to find a proper loop $\gamma=$ that is also heteroclinic, and such that $W^s_\gamma(a)\cap W^u_\gamma(b)=\varnothing$. These proper loops would be the analog of primary intersections. However, such loops need not exist as we saw in Ref. 3. One consequence is that if one fixes a pair of fundamental domains and , then the set of points at which $f^k(U)$ first intersects is not necessarily a union of submanifolds of —in particular the intersection curves may end in the middle of if is not chosen to be properly ''aligned'' with .

To alleviate this problem, we use the intersection index to define the primary intersection of the stable and unstable manifolds of a and b, so that the connected intersection curves are submanifolds:

Definition 3 (Primary Intersection): *Let a and b be hyperbolic fixed points of type A and type B, respectively, whose two-dimensional manifolds intersect. We define the primary intersection of the stable and unstable manifolds as*

$$a,b)=\cup\{\ \cap\ :\ \ \in \mathcal{F}^u \ b),\ \in \mathcal{F}^s \ a),\ \ )=0\}.$$

We assert that is invariant, and is the union of immersed submanifolds of $W^s(a)$ and of $W^u(b)$. Moreover, the intersection of with any fundamental domain is generically a *neat submanifold*. Recall that when M is a manifold with boundary, a set $A\subset M$ is neat in M if

$$\partial A=A\cap\partial M \qquad 4)$$

cf. Ref. 28 for the definition). In other words, the boundary of the submanifold is *nicely placed* in the boundary of the manifold.

For any fixed fundamental domain , the primary intersection does not have to be a neat submanifold of . However, if the intersection of the stable hassert 4u1.2 TDmbl3399.4(bounda method, but that this intersection is generically transverse and along one dimensional curves. Generally the perturbed map takes the form

$$F_\epsilon = F_0 + \epsilon P_1,$$

such that F_ϵ is volume-preserving. We make the simplifying assumption that $P_1(a) = P_1(b) = 0$, so that F_ϵ still has hyperbolic fixed points at a and b. However, stated in terms of P_1, it is not so easy to construct volume-preserving perturbations to F_0. It is easier to let

$$F_\epsilon = (id + \epsilon P) \circ F_0,$$

where I is the identity map. This can always be done since $P \equiv P_1 \circ F_0^{-1}$. In these terms it is easier to construct perturbations that do not destroy the volume-preserving property:

Lemma 3: *Let :13*

such a way that it has two invariant sets which give rise to a saddle connection between two fixed points. Examples similar to these were found by Lomelí[22] and are related to the work of Suris[30,31] on integrable maps. It is interesting to note the map need not have an integral, and therefore, apart from the two invariant sets, typically exhibits chaotic behavior. We finally give an example for which the resulting volume-preserving map has a first integral.

A. Explicit heteroclinic connection

We start with the area-preserving map generated by the Lagrangian generating function $L:\mathbb{R}^2$

The fixed line of this reversor is the x axis, and $S_0(a)=b$. A standard argument[32] implies that points where $W^s(a)$ crosses the x-axis are heteroclinic to b.

Lemma 7 implies that the invariant I can be used to simplify the computation of the Melnikov function. Recall

We show an example of the Melnikov function in Fig. 5, using the coordinates z and   on the fundamental domain. Positive values of $M(\ ,z)$ are shown in shades of red, and negative in shades of blue, and the zero level is shown as the solid black curve. As implied by theorems 6 and 2 the contours of M are neat submanifolds of the fundamental annulus—either closed loops or curves that end on one of the boundaries of the annulus.

In general, since the boundaries of the fundamental domain are γ and $F(\gamma)$, we may use the map F to identify the boundaries the annulus, turning it into a torus. In our example F_0 rigidly rotates the equatorial circle by 2 , so that we merely undo this rotation to perform the identification

$$\{\ +2\ \ ,H(\ )\}\equiv\{\ ,0\}.$$

Since the zeros of M are neat submanifolds, they become closed loops with this identification. Thus the zero contours of M can be classified by their homotopy class, a pair of integers (m, n) that gives the number of times the contour loops around the torus in the z and   directions, respectively. When the zeros of M are nondegenerate, all of the curves must have either the same homotopy class, or the class $(0, 0)$. Each loop has a natural direction, associated with the direction of the crossing of the manifolds. Thus loops with a nontrivial homotopy class must appear in pairs.

In Fig. 5 there are a pair of loops of homotopy class $(1, 0)$, i.e., curves that extend from $z=0$ to $z=H(\ )$ without encircling longitudinally. By the symmetry (20) there are always zeros on the x axis, so $M(0,0)=M(\ ,0)=0$—in the case shown the primary intersection curves through these points have homotopy class $(1, 0)$. Also shown in the figure are a pair of loops of homotopy class $(0, 0)$, i.e., loops that are homotopic to a point. These loops appear in a parameter region corresponding to small   and moderate values of , and disappear either by colliding with a $(1, 0)$ loop, or by shrinking to a point. For example, if we fix $=0.2$, $=0.1$, then for the range 0 rotation about the z axis by 2 is conjugate to one by 2 (1−) under the coordinate transformation →− . There are four distinct regions in Fig. 6, corresponding to loops with homotopy classes (0,1), (1,0), (3,1), and (1,0) with a pair of trivial loops. The parameters for Fig. 5 are near the codimension two, cusp point at (,)≈(0.2,0.15), which corresponds both to the collision of the trivial loops with a (1,0) loop, and their shrinking to a point. Examples of the zero contours of the Melnikov function are shown in Fig. 7, corresponding to the parameter values labeled (a)–(f) in Fig. 6. When is small, the intersection curves are ''equatorial,'' of class (0,1); this corresponds to Fig. 7(d). For small and moderate values of the primary intersections correspond to a pair of (1,0) curves plus a pair of ''bubbles,'' curves with homology class (0,0), as shown of Fig. 7(a). As increases these bubbles disappear, leaving only the (1,0) curves, shown in Figs. 7(b), 7(e), and 7(f). These become increasingly elongated as one approaches the (3,1) bifurcation where they reconnect, as shown in Fig. 7(c), forming a single pair of (3,1) loops.

To compare the actual behavior of the manifolds for the map F_ϵ, we need to choose a reasonably large value of ϵ so that the intersections can be numerically resolved. It is relatively easy to plot the manifold $W^s(a)$ when the pair of stable multipliers at the fixed point have the same magnitude;[25] this is true for our map by (14). In this case one

APPENDIX: PROOF OF THEOREM 6

For each point ξ in the saddle connection $W^s(a) \cap W^u(b)$, there is a neighborhood $_0$ contained in a fundamental domain of the saddle connection, such that all the iterates $F_0^k($ $_0$

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