Volume-preserving maps with an invariant

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Several families of volume-preserving maps on $\mathbb{R}^3$ that have an integral are constructed using techniques due to Suris. We study the dynamics of these maps as the topology of the two-dimensional level sets of the invariant changes.
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Volume-preserving maps arise from the study of the flow of incompressible fluids or magnetic fields. If a volume-preserving map has a continuous symmetry, such as a rotational symmetry, then it has an invariant and the orbits are confined to surfaces. More generally, the orbits could densely cover regions with nonzero volume. Here we construct maps that have an invariant, but no (obvious) symmetry. The dynamics of these maps, while sim-

y,*z*!!. ~1!

grable, area-preserving maps was devised by Suris.[7] He studied maps of the second difference form,

$$x_{t+1} - 2x_t + x_{t-1} = \epsilon F(x_t, \epsilon), \tag{6}$$

which can be thought of as an area-preserving map upon defining the variables $(x,x') = (x_{t-1}, x_t)$. Under the assumptions that F and Φ are analytic and the invariant has the form,

$$\Phi(x,x',\epsilon) = \Phi(x',x,\epsilon) = f_0(x,x') + \epsilon f_1(x,x'), \tag{7}$$

Suris showed there are exactly three possible families. For these cases the corresponding F is rational in x, in trigonometric functions of x, or in exponentials of x, respectively. The three examples of the form (1) that we construct in Sec. II correspond to these three cases; however unlike Suris we have not shown that our solutions are exhaustive.

Other examples of integrable symplectic maps have also been found. Suris' techniques have been used to find higher dimensional, integrable symplectic maps.[8,9] Another technique that gives many examples is to find appropriate discretizations of integrable differential equations; these can be treated with the methods obtained from inverse scattering theory.[10,11] Finally, maps with integrals have been constructed as integration algorithms for differential equations with conserved quantities.[12]

In this paper we will study volume-preserving maps on $\mathbb{R}^3$. Such maps are useful in understanding the motion of passive tracers in fluids[13] and magnetic field line configurations.[14,15] They are also of interest since many phenomena in the two-dimensional case are not yet completely understood in higher dimensions. Such phenomena include transport,[16,17] the breakup of heteroclinic connections,[18,19] and the existence of invariant tori.[20,21] These maps are also important as integrators for incompressible flows; in some cases the maps are constructed to be volume-preserving,[22–25] and in others to preserve the conserved quantities of the flow.[12]

A prominent class of volume-preserving maps that have an invariant are trace maps.[26] Physically, these are obtained from the Schrödinger equation with a quasiperiodic potential.[27] Mathematically, they arise from substitution rules on matrices.[26,28,29] As an example, consider matrices $A,B \in SL(2,\mathbb{R})$, the group of 2×2 matrices with unit determinant. A substitution rule acts on a string of matrices and corresponds to replacements of each occurrence of A and B with strings of these matrices. One of the most studied examples is the Fibonacci substitution rule which corresponds to $A \mapsto B$ and $B \mapsto AB$. The trace map is determined by the action of this substitution on the traces of the matrices. Defining $x = \frac{1}{2}\mathrm{Tr}(A)$, $y = \frac{1}{2}\mathrm{Tr}(B)$, and $z = \frac{1}{2}\mathrm{Tr}(AB)$, then the substitution rule gives $x' = \frac{1}{2}\mathrm{Tr}(B) = y$, $y' = \frac{1}{2}\mathrm{Tr}(AB) = z$, and $z' = \frac{1}{2}\mathrm{Tr}(BAB) = \frac{1}{2}\mathrm{Tr}(AB^2) = -x + 2yz$, where we use the Cayley–Hamilton theorem to simplify the last equation. Thus we obtain the three-dimensional mapping,

$$f(x,y,z) = (y, z, -x + 2yz). \tag{8}$$

This map has a form similar to (1) and is volume-preserving, but the change in sign in the last term means the map is orientation-reversing. All trace maps that arise from invertible substitution rules have the function,

$$\Phi(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1, \tag{9}$$

as an invariant. Roberts calls this function the Fricke–Vogt invariant;[28] it is an example of a group theoretic invariant called a character. In this case, (9) arises from the trace of the wor iTTion-7r0 9 0.34 096ic (xyz)Tj /F6t;2

orbits and their bifurcations. The existence of the invariant implies that these orbits come in one parameter families that are transverse to the level sets M_m, except at bifurcation points. We will also show some numerical examples of the dynamics.

One reason for studying maps of the form ~1! is that they are volume-preserving for arbitrary *F*. Moreover, this form also arises quite generally for the case of quadratic automorphisms. Accord utomorRef.9etn3(th30,9etn3(thany9etn3(such9etn3(ps)(utomorh)-466.3(th)-402omonon]TJ T* [(phtrivial,298.985olume-p2
~1!

condition. In such cases, f_0 is a polynomial of degree at most two in each variable. Since f_0 is even and invariant under cyclic permutation, we have

$$f_0 \sim x,y,z! = a_0 \sim x^2+y^2+z^2!$$

$$+b_0 \sim x^2y^2+y^2z^2+z^2x^2!+c_0\, x^2y^2z^2, \quad \sim 23!$$

up to additive constants. From ~20! it follows that

$$\sim f_1 \sim x,y,u!+ f_1 \sim x,y,-u!!F|_0$$

$$=-2\, f_0 \sim x,y,u! \qquad _0 \quad \mathrm{X}$$

$$\mathsf{F}_{e}\sim x,y,z\,! = \mathsf{F}_{e}\sim y,z,-x+2eF\sim y,z,e\,!\,!.$$

Then the symmetry ansatz, ~15!, should be replaced by

$$\mathsf{F}_{e}\sim x,y,z\,! = \mathsf{F}_{e}\sim y,z,x\,!.$$

so that the remaining multipliers satisfy $\lambda_1+\lambda_2=t-1$. These two multipliers correspond to the map restricted to the invariant surface when the orbit is not in the critical set of Φ. Thus if we consider the restricted map, the periodic orbit is elliptic if $-1<t<3$, hyperbolic with reflection if $t<-1$, and hyperbolic if $t>3$. If $t=-1$, the restricted map has a double multiplier at -1, so that a period-doubling is expected. In the case $t=3$, $\lambda=1$ is a double eigenvalue and a saddle-center bifurcation is expected. More generally suppose that φ_0 is not a critical point of Φ and that at this point a curve of period n points, intersects a period $k\cdot n$ curve. Then the linearization of f^n at φ_0 must have a k^{th} root of unity as eigenvalue so that $t=1+2\cos(2\pi(m/k))$ for some integer m.

B. Invariant surfaces

The topology of the level sets of the invariant (4),

$$\Phi(x,y,z)=x^2+y^2+z^2+\alpha(xy+yz-zx)$$
$$+\gamma(x^2y^2+y^2z^2+z^2x^2)+\beta(x^2yz+z^2xy$$
$$-y^2zx)+\delta x^2y^2z^2,$$

depends significantly on the parameters α, β, and γ

Solutions only exist when $a < -2$ or $a > 18g-2$. When $a < -2$, ~34! represents one closed curve. For $a > 18g-2$, ~34! corresponds to two closed curves lying on each side of $y = x + z$.

For the special case $d = 0$, we can relatively easily classify the possible topologies of the sets M_m. In this case there is at most one critical orbit in each of the classes described above. We label the critical levels corresponding to the ~**Ci**! by m_i. When they exist, the critical levels appear in the order

$$m_2 < m_3 = m_4 < m_0 = 0$$

while m_1 may vary in the ordering.

When $a < -2$ there are two period six orbits. The first, ~**C2**!, is born at m_2 which has an expression—arising from the discriminant of ~33!—that is too long to display. The second period six orbit ~**C3**! is born at

$$m_3 = -\frac{\sim 2 + a!^2}{4g}.$$

The critical circle ~**C4**! exists when $b = -2g$ and $a < -2$. In this case $m_4 = m_3 = m_2$, and the orbits ~**C2**! and ~**C3**! become part of the critical curve. Finally the period two critical orbit arises only when $(1-a)/(g-b) < 0$ at the level

$$m_1 = \frac{3\sim a - 1!^2}{4\sim b - g!}.$$

In the special case $a = 1, g = b$

Even though the fixed point at the origin is critical, it always has one unit multiplier since it lies on the curve of fixed points. It is elliptic when $-3<\alpha<1$.

Period two points have the form $(x,y,x)\rightarrow(y,x,y)$, where x and y lie on the curve

$$\gamma(x^2+y^2)+$$

~**C1**! $m_1 = -(a-1)^2/4g$, corresponding to the critical period two orbit.

~**C0**! $m_0 = 0$ corresponding to the critical point at the origin.

The fixed points lie on the line (x,x,x). If $a < -3$ there are no fixed points on M_m until

The right panel shows the dynamics for $m=1.0$. Here one can see a prominent island ~purple! enclosing one of the elliptic fixed points. Again the invariant surface is divided into two large chaotic domains. For $m>1.1$ the large invariant circles have been destroyed, and the two chaotic zones are joined. There are prominent elliptic regions until after the fixed point orbit period doubles @from ~39!, $m=3.75$]. For larger m the dynamics appears nearly uniformly chaotic; however, amongst the chaotic orbits are the islands surrounding the two elliptic period three orbits. These become more visible for large m.

The orbits for the case corresponding to Fig. 4 are shown in Fig. 7. When $\frac{1}{4}<m<0$, the orbits that lie on the pair of spheres enclosing the critical period two orbit are predominantly regular, as can be seen in the left panel. As m approaches 0, the chaotic regions grow, and they dominate the critical surface, $m=0$, as seen in the middle panel. There are also large islands surrounding the elliptic period two orbits at this level. Near $m=0.42$ a family of invariant circles appears that divides the chaotic region into two parts, as can be seen in the right panel. These circles are destroyed by $m=1.8$, and as before, apart from the elliptic period three orbits, the dynamics is largely chaotic as m becomes large and the invariant surface acquires its hourglass shape.

As a final example, we consider the parameters corresponding to Fig. 5. For this case, orbits on compact components of six level sets are shown in Fig. 8. In the top-left panel, $m<-1$, and the orbits lie on a family of six spheres enclosing the ~**C2**! orbit. In the next panel, these spheres have joined at the ~**C3**! orbit, and the dynamics appears uniformly chaotic. In the top-right panel, $m=0$, the torus pinches at the origin. The red and black orbits encircle the elliptic fixed points. Also shown are green and yellow orbits that are associated with two elliptic period five orbits. For larger m, as shown across the bottom row in Fig. 8, the islands around the elliptic fixed points remain prominent. Also visible are two elliptic period four orbits ~light blue and green! in the bottom-middle panel at $m=5$. Apart from these islands, which persist on the unbounded components for $m>18.75$, the dynamics on these sets appears to be largely unbounded.

IV. CONCLUSIONS

We have used the methods of Suris to find several families of volume preserving maps on $\mathbb{R}^3$ that have an invariant. Unlike Suris, our solutions do not appear to be exhaustive. It would be interesting to obtain such a classification. We have not found any polynomial maps that have an invariant beyond the trace maps, ~8!–~10!. It may be that there are no polynomial, volume-preserving maps which have an invariant that satisfies the conditions ~15!–~16!; our results show this is true when F is a homogeneous quadratic function.

Both topologically and dynamically our maps are richer than the well-known trace maps. We do not know if there is a set of parameter values for which our maps are ‘‘completely chaotic’’ on an invariant surface; this was one of the prominent features of trace maps, which are semiconjugate to an Anosov system on the tetrahedral critical level set of the Fricke–Vogt invariant.

In the future it would be interesting to investigate the dynamics of these maps composed with a small perturbation that destroys the invariance of F. Is the transport between level sets more efficient when the dynamics on the surface is chaotic?

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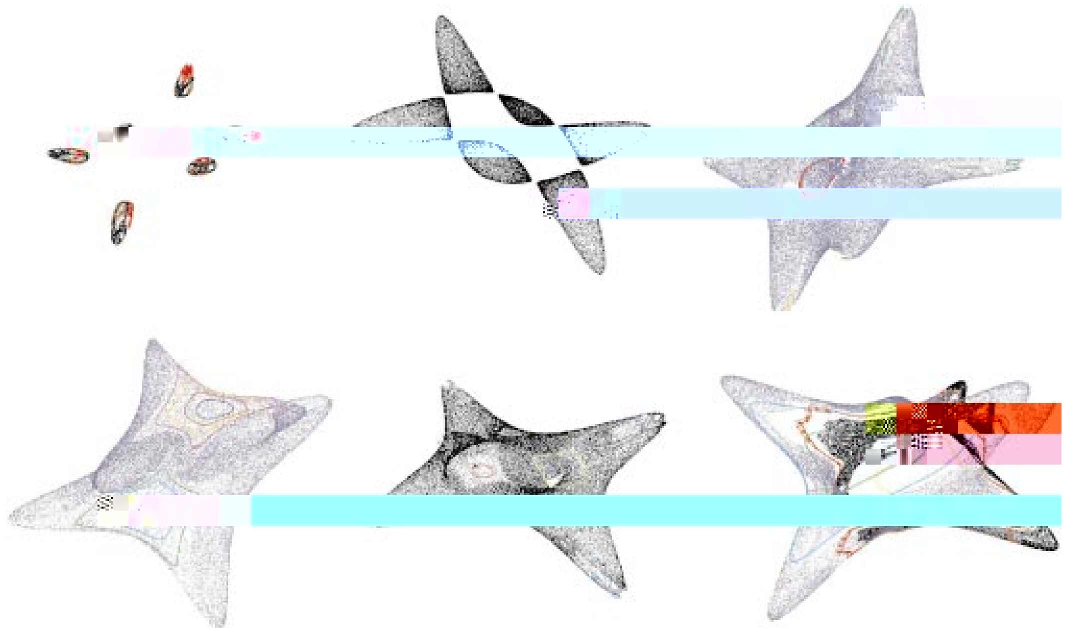

FIG. 8. ~Color! Orbits of ~1!

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