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Figure 2. The Change and Find menus for *StdMap*. In this tutorial, we denote a menu selection using the → symbol, thus `Change → Show Axes` indicates that we are selecting the sixth item in the change menu. This item is checked since the axes are currently shown. Selecting it will toggle the display of the axes in the plot window. Your selections in this menu will be remembered the next time you start *StdMap*.

Figure 3. Chaotic orbits appear to densely cover a fat fractal like this set generated by iteration of a single initial condition for $k = 1.0$ (left pane) and $k = 2.0$ (right pane).

proven [9,10]. How long do you have to iterate until the trace of the orbit settles down and no new pixels are filled? If you have a monitor with many pixels, this time can be very long indeed [11]. If we change the parameter value by selecting `Change → Map Parameters...`, and typing the value 2.0 for k in the `Parameter Dialog` window, the large chaotic region has the form shown in figure 3.

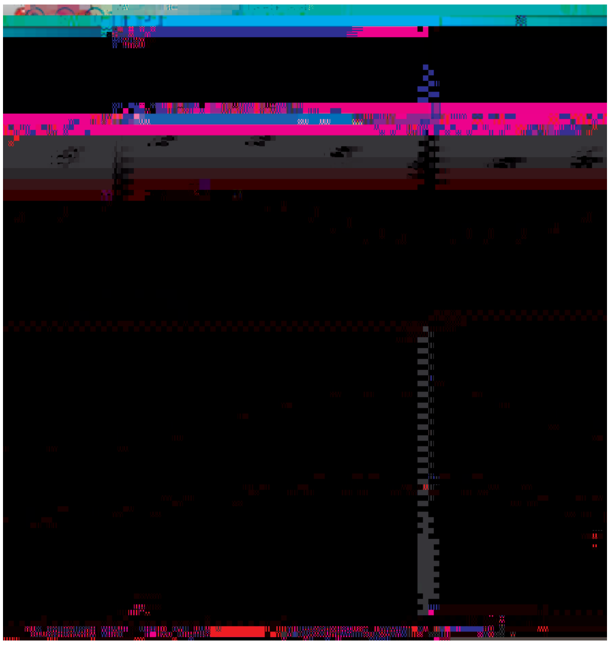

Figure 4. Iterating the standard map one step at a time for $k = 0.3$.

Often iteration in *StdMap* is too quick to really see what is going on; there are two ways to slow it down. One is to change the iteration mode in the `Find` menu. In particular `Find` → `Single Step (spacebar)` stops the iteration and moves the point forward (now drawn as a small square instead of a pixel) only when you hit the spacebar, recall figure 2. Let us also change the parameter of the map to something smaller so that there is less chaos, choose `Change` → `Map Parameters...`, and type the value 0.3 for k in the `Parameter Dialog` window. Note that the current value of k is displayed at the bottom of the plot window. Now when you click on an initial point, and repeatedly hit the spacebar to iterate step-by-step, you will see a portrait like that shown in figure 4 [12].

Iterating one step at a time reinforces the fact that maps are dynamical systems with discrete time. Another key feature of the map (1) is that the horizontal distance between successive iterations grows with the momentum value. Mathematically this is an example of the *twist* condition,

$$\tau = \frac{\partial x}{\partial y} = 0. \tag{2}$$

For the standard map, $\tau = 1$, and so it twists to the right. Perhaps a better way to visualize twist is to iterate a curve of initial conditions instead of a single point. You can do this in *StdMap* by selecting `Find` → `Curve...` or typing ⌘-J, recall figure 2. This will open the `curve dialog`, as shown in figure 5. There are five types of curves that you can iterate, and you can select one by clicking in one of the boxes. For this demonstration, click on the middle box, which selects the line type.

4. The onset of chaos

The standard map is *integrable* when $k = 0$. Indeed for this value of k, the momentum is an invariant, and all the orbits lie on horizontal curves. More generally, an area-preserving map is integrable [1,16] if there exists a non-constant function $I(x, y)$ such that

$$I \circ f = I.$$

There are several other integrable maps that can be studied in

Figure 6. Overlay showing the evolution of two orbits beginning at $k = 0$ as k is incremented. In the left pane, an initial circle $y \approx 0.02$ evolves to the chaotic trajectory near a period-4 saddle when $k = 2.25$. In the right pane, the initial circle at $y \approx$

when $k = 2.0$. This particular period-5 orbit is a saddle [21]. We can determine this

Turning on the residue calculation for the $(2, 5)$ orbit at $k = 2.0$ and finding the orbit again, gives $R = -11.88$, confirming that this orbit is also a regular saddle. Indeed, if you start iterating at this orbit, by selecting `Find → Continuously` (which automatically fills in the initial condition of the last orbit) you will find that the instability leads to numerical error: the iteration falls off the periodic orbit and rapidly covers the chaotic fat fractal shown in figure 3. Another way to see this is to find the $(2, 5)$ saddle, `Find → Periodic Orbit...`, and then immediately select `Find → Single Step` to start single-step iteration at the point (6). As you iterate with the spacebar, you will see that the orbit visibly deviates from the periodic orbit after about 40 steps. For the given residue, the unstable eigenvalue of M is $\lambda_+ = 49.5$, so that any error grows by a factor of $\lambda_+^{N/5} \approx 4 \times 10^{13}$ by $N = 40$; thus given the inevitable truncation error of double-precision floating point computations, it is not surprising that the orbit is lost.

There are better ways of investigating the properties of hyperbolic saddles, as we describe next.

5. Stable manifolds

The stable and unstable sets, W^s and W^u, of an invariant set Λ are defined as

$$W^s_\Lambda \equiv \{(x, y) : f^t(x, y) \to \Lambda \text{ as } t \to \infty\},$$
$$W^u_\Lambda \equiv \{(x, y) : f^t(x, y) \to \Lambda \text{ as } t \to -\infty\}. \qquad (9)$$

If Λ is hyperbolic, then these sets are smooth submanifolds that are tangent to the eigenspace of the linearization of the map at Λ [24,25]. For example, at the hyperbolic fixed point $(0.5, 0)$, the linearization (7) has eigenvectors

$$v_\pm = \begin{pmatrix} 1 \\ 1 - \lambda_\pm \end{pmatrix}$$

with $\lambda_\pm = \frac{1}{2}(2 + k \pm \sqrt{k(k + 4)})$. When $k > 0$ the $+$-eigenvector corresponds to the unstable direction, and since $\lambda_- < 1$, it has a positive slope. The stable direction has negative slope. The stable manifold theorem implies that $W^{u,s}$ are smooth curves that start at $(0.5, 0)$ with slopes $1 - \lambda_\pm$.

To find these, *StdMap* uses an algorithm suggested by Hobson [26]. Selec89.6([)28001.0009.96009.9653.15-475

W^u for the (p, q) orbit. If these points are more than one pixel apart the program inserts a new point between them using linear interpolation.

Hitting spacebar again causes all of the points on the fundamental segment to be iterated q more times, and more points to be interpolated as needed. The result is a growing curve forming the unstable manifold. The color scheme follows a convention suggested by Bob Easton: red – for unstable – represents the blood moving away from the heart, and blue – for stable – represents the blood returning. The fundamental segment is stored in the large array and iteration will stop if this array fills up. Examples of unstable manifolds for $k = 1.0$ and 2.0 are shown in figure 7. Though W^u begins life as a nearly straight line along v_+, it rather quickly

$$\begin{aligned} \mathrm{Fix}(S_1) &= \{(0, y)\}, \\ \mathrm{Fix}(S_2) &= \{(x, y) : x = y\}. \end{aligned} \tag{11}$$

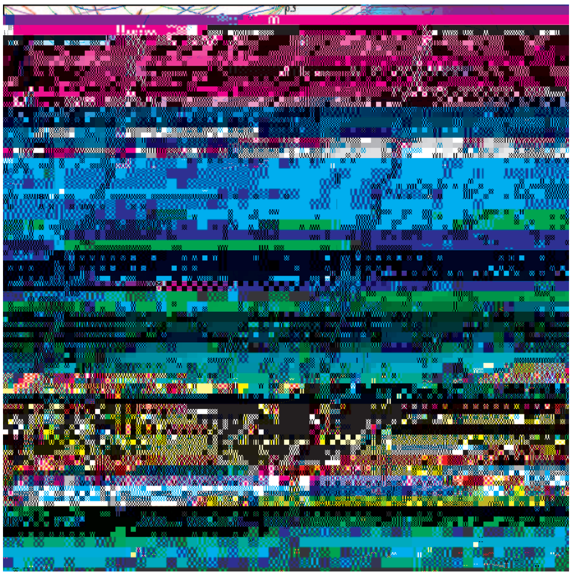

Figure 8. Four forward and four backward iterates of the symmetry lines for (1) at $k = 1.3$ and several orbits near some elliptic, symmetric orbits.

The generalized standard map also has the inversion

$$I(x, y) = (-x, -y)$$

as a symmetry when F is odd. This symmetry can be used in *StdMap* to 'mod'

Table 1. Symmetry lines containing (p, q), symmetric periodic orbits for a reversible map with reversor S_1 and discrete rotation symmetry R. Here $S_2 = f \circ S_1$, $S_3 = S_1 \circ R$ and $S_4 = S_2 \circ R$q2

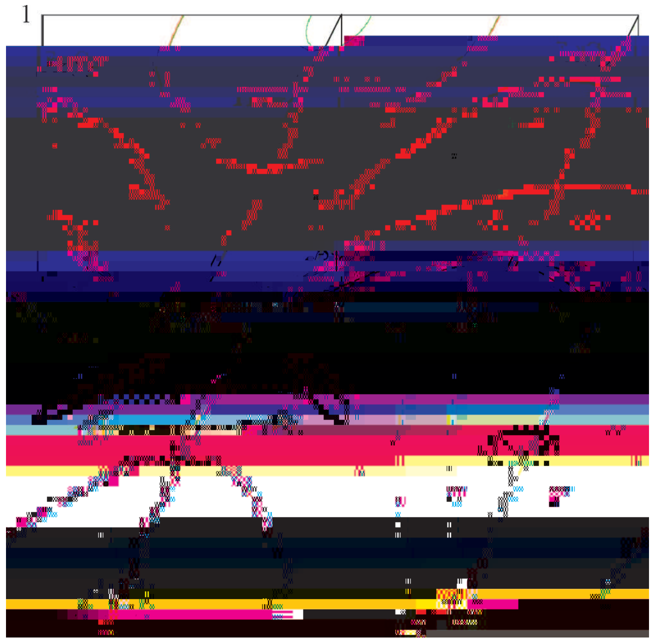

Figure 9. Symmetry lines and some symmetric orbits for the standard map with $k = 1.0$.

The first two entries in the `symmetry` menu, `Minimizing` and `Minimax` refer to the action-minimizing and minimax orbits of Aubry–Mather theory, see [1] for a review. The minimizing orbit is always hyperbolic [39] and corresponds to the second orbit above. The minimax orbit has positive residue and, when k is positive and small enough, it is elliptic. It has a point on the dominant symmetry line.

7. The critical golden circle

The $(0, 1)$ resonance surrounding the elliptic fixed point is enclosed by a connected chaotic region generated by the unstable manifold of the $(0, 1)$ saddle orbit. Chirikov noticed that this chaotic region appears to be bounded when $|k| \lesssim 1$ and unbounded for larger values of $|k|$ [40]. One way to see this is to look for *climbing* orbits, that is orbits that move from $y = a$ to $y = b$ for $a < b$. *StdMap* provides a convenient way to do this experiment: select `Find → Transit Time...`. The program then asks you to drag the mouse over a rectangle, R_i, in which the initial conditions will be selected – for example, choose a rectangle near the fixed point (0.

Visual explorations of dynamics

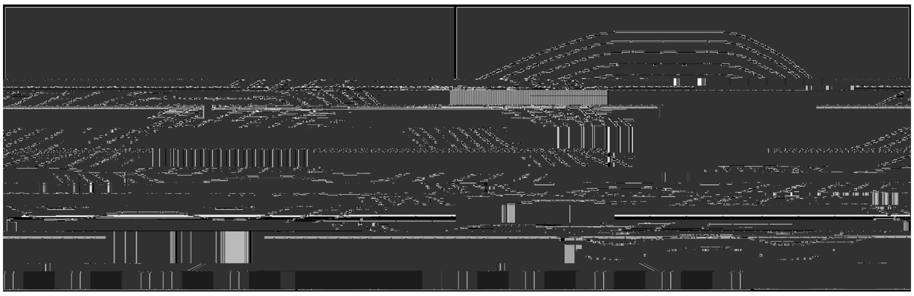

Figure 11. Golden mean invariant circle and approximations to the Cantori for the standard map at $k = k_{cr}$, 0.974, 0

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Acknowledgement

this menu will give the elliptic (2,5) orbit. The symmetries of periodic orbits will be discussed in §6.

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[23] J M Greene, R S MacKay, F Vivaldi and M J Feigenbaum, *Physica* **D3**, 468 (1981)

[24] C Robinson, Dynamical Systems: Stability, Symbolic Dynamics and Chaos, *Studies in Advanced Mathematics*, 2nd edition (CRC Press, Boca Raton, Fla., 1999)

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[27] There is also a pop-up menu for selecting the symmetry of the orbit. Leave this set at the default `minimizing` as this corresponds to the hyperbolic saddle.

[28] We allocate an array that is no more than half the physical memory in your computer and no larger than needed to contain up to 10 times the number of pixels in the plot window.

[29] V F Lazutkin, I G Schachmacnnkki and M B Tabanov, *Physica* **D40**, 235 (1989)

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[56] It would be best, of course, for me to add a parser to *StdMap* so that arbitrary