

3. Let $N \geq 0$ be an unknown integer and let X_1, \dots, X_n be a random sample from the distribution with

$$P(X_i = k) = \begin{cases} \frac{1}{2N+1} & ; k = 0; 1; 2; \dots; N; \\ 0 & ; \text{otherwise:} \end{cases}$$

- (a) Show that $M = \max\{X_1, \dots, X_n\}$ is a sufficient statistic for N .
- (b) Show that M is a complete statistic for N .
- (c) Determine the uniformly minimum variance unbiased estimator (UMVUE) for N . Simplify your answer!
Hint for part (c). Determine constants $a; b; c$ such that $aM + b \mathbb{I}[X_1 = 0] + c$ is unbiased for N , where $\mathbb{I}[X_1 = 0]$ is notation for the indicator function of the event $[X_1 = 0]$.
4. Suppose that X_1 is a random sample of size 1 from the distribution with probability density function

$$f(x; \theta) = \begin{cases} \frac{1}{2} & ; \text{if } 0 < x < \theta; \\ \frac{1}{2x^2} & ; \text{if } x > \theta; \\ 0 & ; \text{otherwise:} \end{cases}$$

- For which values of $0 < \theta < 1$ does $H_0 : \theta = 1$ versus $H_1 : \theta < 1$ admit a unique uniformly most powerful (UMP) test of size α ? Specify the rejection region associated with each of those tests.
5. Let $0 < \lambda < \mu$ be real constants and consider an M/M/1 queueing system with arrival rate λ and service rate μ , which is initially empty. Suppose that the server is turned ON and OFF according to the following rules:
- it remains OFF as long as the number of customers in the system is less than 2,
 - it is turned ON as soon as the number of customers in the system becomes 2 and then remains ON until completely emptying the system of customers.
- Based on the above, please respond:
- (a) Model the number of customer in the system as a continuous time Markov chain with state space $\{0; 1'; 1; 2; 3; \dots\}$, where $1'$ represents the configuration of having exactly one customer in the system while the server is OFF, and state 1 represents the same but while the server is ON. Represent the rate transition matrix of the chain as a directed graph with weighted edges.
- (b) Show that this system has a stationary distribution and determine it explicitly.
- (c) After a long time of operation, what is the probability that a new customer encounters the queue empty?