

Department of Applied Mathematics  
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION  
August 2020

Instructions:

Do two of three problems in each section (Prob and Stat).  
Place an **X** on the lines next to the problem numbers  
that you are **NOT** submitting for grading.

Prob  
1. \_\_\_\_  
2. \_\_\_\_  
3. \_\_\_\_

Do not write your name anywhere on this exam.  
You will be identified only by your student number.  
Write this number **on each page** submitted for grading.  
Show all relevant work!

Stat  
4. \_\_\_\_  
5. \_\_\_\_  
6. \_\_\_\_  
Total \_\_\_\_

Student Number \_\_\_\_\_

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## Probability Section

### Problem 1.

- (a) Let  $X$  be a non-negative continuous random variable with cdf  $F$ . Show that

$$E \frac{1}{1+X} = \int_0^{\infty} \frac{F(x)}{(1+x)^2} dx$$

- (b) Let  $X_1; X_2; \dots; X_n$  be a random sample from the exponential distribution with rate 1. Define

$$Y_n = X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3 + \dots + \frac{1}{n}X_n;$$

Find expressions for the moment generating functions for  $X_{(n)}$  and  $Y_n$ . (Your expressions may contain sums or products but may not contain integrals.)

- (c) Compare the two moment generating functions for  $n = 1; 2$ , and  $3$ . Assuming that the pattern you see continues for all  $n \geq 1$ , what can you say about the distributions of  $X_{(n)}$  and  $Y_n$  as they relate to each other?
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**Problem 2.**

Let  $\{X_n\}_{n=1}^\infty$  be a sequence of iid random variables from the Poisson distribution with parameter  $\lambda$ . Let  $Y_n = X_n X_{2n}$  for  $n \geq 1$  and consider the  $n$ th partial sum

$$S_n = Y_1 + Y_2 + \dots + Y_n$$

- (a) Find  $E[S_n]$ .
- (b) Find a constant  $C$ , which may depend on  $\lambda$  but which may not depend on  $n$ , such that

$$\text{Var}[S_n] \leq Cn \quad \forall n \geq 1$$

- (c) Find a sequence of real numbers  $\{a_n\}$  such that

$$\frac{S_n}{a_n} \xrightarrow{P} 1$$

(Your  $a_n$  may depend on  $\lambda$ .)

**Problem 3.**

In a disease outbreak, there are three different states of an individual: the first state is "susceptible" (denote by  $s$ ), the second is "infected" (denoted by  $i$ ), and the third is "recovered" (denoted by  $r$ ). The state of an individual at time  $t \geq 0$ ,  $X(t)$ , is modeled as a continuous-time Markov chain with the infinitesimal generator (or rate matrix)

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ \beta & -\beta & 0 \\ 0 & 0 & -\gamma \end{pmatrix} \tag{1}$$

for some  $\lambda, \beta, \gamma > 0$  with  $\lambda < \beta$ . We assume that  $X(0) = s$ .

- (a) Consider the first infection time  $\tau_i := \inf\{t > 0 : X(t) = i\}$ : Given  $t > 0$ , find the probability  $P(\tau_i > t)$ .
- (b) Given  $t > 0$ , find the probability that  $X(t) = i$  and the state  $r$  has not yet been visited, i.e.

$$P(X(t) = i \text{ and } X(u) \neq r; \forall u \in [0, t])$$

- (c) Given  $t > 0$ , what is the probability that the individual get infected three times during the period  $[0, t]$ ?
- (d) Suppose that there are  $N > 0$  individuals in a population. Each individual is susceptible at time 0, and subject to the spread of the disease as in (1) *independently of other individuals*. As  $t \rightarrow \infty$ , what are the limiting fractions of population that are susceptible, infected, and recovered?

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## Statistics Section

### Problem 4.

Consider  $X_1; X_2; \dots; X_n$  where  $X_i$  is exponentially distributed with mean  $= \theta_i$ . Let  $Y_1; Y_2; \dots; Y_n$  be exponential random variables with  $E[Y_i] = \theta_i$ . Assume that the  $X$ 's and  $Y$ 's are all mutually independent.

In this problem, the parameters  $\theta_1; \theta_2; \dots; \theta_n$  are all positive and unknown.

- (a) Find the maximum likelihood estimator (MLE) of  $\theta$ .

For parts (b) and (c), assume that  $\theta_1; \theta_2; \dots; \theta_n$  are known.

- (b) Find the MLE for  $\theta$  and the UMVUE (uniformly minimum variance unbiased estimator) for  $\theta$ .  
(c) Compute the relative efficiency of your estimators from part (b). What can you say as  $n \rightarrow \infty$ ?

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### Problem 5.

Suppose that  $X$  and  $Y$  are iid  $N(0; 1)$  random variables. It is well known that  $X^2$  and  $Y^2$  each have a  $\chi^2(1)$  distribution.

- (a) Let  $W = \min(X; Y)$ . Show that  $W^2 \sim \chi^2(1)$ .  
(b) Now suppose that  $X$  and  $Y$  are iid  $N(\mu; \sigma^2)$  random variables with  $\mu$  known and  $\sigma^2$  unknown. Use part (a) to derive a  $100(1 - \alpha)\%$  confidence interval for  $\sigma^2$  based on the statistic  $W = \min(X; Y)$ .

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### Problem 6.

Suppose that we have a random sample,  $X_1; X_2; \dots; X_n$  from the distribution with pdf

$$f(x; \theta) = \frac{1}{6} x^2 e^{-x/\theta} I_{(0; \infty)}(x)$$

- (a) Find the best (most powerful) test of size  $\alpha$  of  $H_0: \theta = \theta_0$  versus  $H_1: \theta = \theta_1$ , assuming that  $\theta_1 > \theta_0$ . Give your answer in terms of a chi-squared critical value.  
(b) Is your test uniformly most powerful (UMP) for the alternative hypothesis  $H_1: \theta > \theta_0$ ? Explain.  
(c) Is your test uniformly most powerful (UMP) for the alternative hypothesis  $H_1: \theta \neq \theta_0$ ? Explain.  
(d) Derive an approximate large-sample generalized likelihood ratio test (GLRT) of size  $\alpha$  for the hypotheses in parts (b) and (c) if your test was not a UMP test. (Note: Depending on how you answered (b) and (c), you may have nothing to do here, you may have one test to do, or you may have 2 tests to do.)
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