A draft of INRIA lectures, May 1991

Wavelets, Multiresolution Analysis and Fast Numerical Algorithms

Bey n

n od e on

e e ec e e de o ed o f n e c o nd e e on o n e of po n e on c An y e ep e en on of de c of ope o n e e fo e p e C de on Zy nd o p do d en ope o y e e ed e od fo e "co p e on e con e on o p fo e p y of e ep e en on con eq ence of e oc z on of e e n o p ce nd e n e do n n dd on e e on c e of e e e e p n on n o n e c en o n z on of n fo on on en e nd n e c on e een d en ne o n e c n o n z on of o ne nd non ne n fo on een po e f oo on c An y y efe ed o L e ood ey nd C de on Zy nd eo e e e nd no ppe o e n eq y po e f oo n N e c An y

n pp c on fo e p e n e p oce n nd n c e e on e od e e de e oped n c fo e fo e n p oce n o ed on e o e n fo A ec n q e of nd cod n e e c q d e o e M n od ced n 7 c e e n oc on e o d o o e p e n y nd e f o ed e y of e o ono of f nc on p ce n f c e e e e e q c on on y o o c o ce e o e nd e o e e o e n pode n e of e p ce e n e o e f eq ency oc z on e efo e e e d o e y e de e o e e e n n n e n e of o o ono con o e o c z on n e p ce e n e do n e e n e c nd o pp ed e d c n ed n e de e op en of o ono of e e nd e no on of M e on An y 7 e e e ny ne con c on of o ono con o e oc z on n e p ce e n e do n no y 7

n N e c An y ny n ed en of C de on Zy nd eo y e e ed n e M po e o fo co p n po en n e c on e 4 e

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M po e I o eq e o de N ope on o co p e e

$$p_{\mathbf{j}} = -\sum_{\mathbf{i} \neq \mathbf{j}} \frac{q_{\mathbf{i}} q_{\mathbf{j}}}{| \;\; \mathbf{i}}$$

e e od y e e ed de ce fo ed c n p d . en eq on o
p e ne y e fo e co of n n e en y cond on n e of e e n
ce f n e d of n e d . ence o n e e e en ep e en on e e e ep
e en on of e de e n e e en p e od c on

e n y e

II.1 Multiresolution analysis.

e e de n on of e e o on n y no on n od ced y Meye , nd M , c p e e e en f ee n en e of e en o o of ene y e — o e ne e n e d of e e en e

$$V_n \subset \subset V \subset V \subset V \quad V \subset L \; R^d$$

n n e c e z on e p ce **V** n e d en on

II.2 The Haar basis

o on e e on y e p e of e e o on n y fy n De n on Cond on e Co p n n o. p e of e o e de e n e ec e nd p o de f p o o ype fo n e c e pe en on f d — en $j;k$ — $-j=$ $-j$ — $\in$ **Z** fo ed y e d on nd n on of n e f nc on

$$— — \begin{cases} \text{fo} \\ \text{fo} \quad \leq \\ \text{e e e} \end{cases} \qquad 7$$

n c — e e e c c e c f nc on of e n e o e c $j;k$ — $-j=$ $-j$ — $\in$ **Z** e of V_j nd $j;k$ — $-j=$ $-j$ — $\in$ **Z** e of W_j

e deco po on of f nc on n o n o de N p oced e en N — n p e of f nc on c y fo p c y e o of e of ed e e of f on n e of en $-n$

$$_k — _{n=} \int_{-n k}^{-n k+)} f \; d$$

e o n coe c en

$$d_k^{j+} — \frac{}{\sqrt{}} \; {}^{j}_{k-} - {}^{j}_{k}$$

nd e e

$$^{j+}_{k} — \frac{}{\sqrt{}} \; {}^{j}_{k-} \quad {}^{j}_{k}$$

fo — n — nd — $n-j-$ — e y o ee e n e o e of coe c en d_k^j $_k^j$ n eq e N — dd on nd N p c on n o d en on e e e o n — 96 1 d 3.64 -14.4 Td f 5.7564 0 Td 1f d d en o e nd d fo e econd de ned y e e of ee nd of f nc on ppo ed on q e $_{j;k}$ $_{j;k'}$ y $_{j;k}$ $_{j;k'}$ y nd $_{j;k}$ $_{j;k'}$ y e e e c c e c f nc on of e n e nd $_{j;k}$ $:=$ $^{-j=}$ $^{-j}$ $=$ ep e en n n ope o n e d o e non nd d fo e e no o y eco e c e e

By con de n n n e ope o

$$f \; := \int y \; f \; y \; dy$$

nd e p nd n e ne n o d en on e nd fo C de on Zy nd nd p e do d en ope o e dec y of en e f nc on of e d nce f o e d on f e n e e ep e en on n n e o n e ne e e c of ope o e en y n e o d on e ne e oo y f o e d on o e p e e ne y of C de on Zy nd ope o fy e e e

$$| \quad y | \leq \frac{}{|\; - y|}$$

$$| \partial^{M}_{x} \quad y | \; | \partial^{M}_{y} \quad y | \leq \frac{C_M}{|\; - y|^{\;+M}}$$

fo o e $M \geq$ Le $M :=$ n nd con de

$$^{j}_{kk'} := \int\int \quad y \quad _{j;k} \quad _{j;k'} \; y \; d \; dy$$

e e e e e d nce e een $| \; - \; '| \geq$ nce

$$\int \; _{j;k} \; d := \qquad \qquad (4)$$

e e

$$| \qquad y \qquad f \; r \quad x$$

e on e n n e dec y n c en o e co p n n p c c o e f e dec y nece y o e f nc on e e n n o en e n n o en e e pon e fo n n p c c o e con o n e con n n e co p e y e e of e f o

II.3 Orthonormal bases of compactly supported wavelets

e q e on of e e ence of e on n y oo f nc on of Cond on e no e ed n e con c on of e o ono e e ene z n f nc on y o e . , nd Meye . , e con de econd e o ol on y of $\{ \quad - \quad \}_{\mathbf{k}\in\mathbf{Z}}$ p e

$$\mathbf{k} \quad \overline{\quad} \int_{-\infty}^{+\infty} \quad - \quad d \quad \overline{\quad} \int_{-\infty}^{+\infty} | \quad | \; \mathrm{e}^{-\mathrm{i}\mathbf{k}} \; d$$

nd e efo e

$$\mathbf{k} \quad \overline{\quad} \int \sum_{\mathbf{l}\in\mathbf{Z}} | \quad \quad | \; \mathrm{e}^{-\mathrm{i}\mathbf{k}} \; d$$

nd

$$\sum_{\mathbf{l}\in\mathbf{Z}} | \quad \quad | \quad \overline{\quad}$$

n e o n

$$\sum_{\mathbf{l}\in\mathbf{Z}} |$$

no d c o o ee e . , . , . . , on e c ed e ; ∈ **Z** e
e e { $_{j;k}$ — $^{-j=}$ $^{-j}$ — }$_{k \in \mathbf{Z}}$ fo n o ono of $\mathbf{W_j}$
e fo o n e D ec e , c c e ze ono e c po yno
o on of c co e pond o e o ono e of co p c y ppo ed
e e n n o en

Lemma II.1 Any trigonometric polynomial solution of (2.26) is of the form

h i M — ½ e^i e^i

where $M \geq$ **is the number of vanishing moments, and where is a polynomial, such that**

| e^i | —P n ½ n^M ½ ½ co

where

P y — Σ_k^{M−} (M −) y^k e

and is an odd polynomial, such that

≤ P y y^M ½ − d f

e e $_{\mathbf{k}}^{\mathbf{j}}$ nd $d_{\mathbf{k}}^{\mathbf{j}}$ y e e ed pe od c eq ence e pe od $^{\mathbf{n}-\mathbf{j}}$ Co p n
nd ed y e py d e e

$$\begin{array}{ccccccc} \{\ _{\mathbf{k}}\} & \longrightarrow & \{\ _{\mathbf{k}}\} & \longrightarrow & \{\ _{\mathbf{k}}\} & \longrightarrow & \{\ _{\mathbf{k}}\} \quad \cdots \\ & \searrow & & \searrow & & \searrow & \\ & & \{d_{\mathbf{k}}\} & & \{\tilde{d} & & \end{array}$$

e en de ne $f_{\mathbf{m}} := f_{\mathbf{m}} - \mathbf{m} \cdot f$ e e $\mathbf{m}$ c o en o $\langle f_{\mathbf{m}} \ \mathbf{M} \rangle :=$ fo
$:=$ M c e n e de ed o o on y o $\mathbf{M}$ y e con n e o

e e $\mathbf{V}_j^M$ en n e nd e p ce $\mathbf{W}_j^{M;}$ e o on co p e en of $\mathbf{V}_j^{M;}$ n $\mathbf{V}_{j-}^{M;}$

$$\mathbf{V}_{j-}^{M;} \;-\; \mathbf{V}_j^{M;} \quad M \quad \mathbf{W}_j^{M;}$$

e p ce $\mathbf{W}^{M;}$ nned y e o ono

$$\{ \;\mathbf{i} \quad \mathbf{l}\, y \quad \mathbf{i} \quad \mathbf{l}\, y \quad \mathbf{i} \quad \mathbf{l}\, y \quad - \quad M\}$$

e e $\{\ {}_{\mathbf{m}}\}_{\mathbf{m}}^{\mathbf{m}\ M}$ n o ono fo $\mathbf{V}^M$ c e e en on e M e e en n n o en e o on o e po yno $\mathbf{i}\, y^{\mathbf{l}}$ $-$ $M -$

e p ce $\mathbf{W}_j^{M;}$ nned y d on nd n on of e f nc on of $\mathbf{W}^{M;}$ nd e of $\mathbf{L}$ con of e f nc on nd e o o de po yno $\mathbf{i}\, y^{\mathbf{l}}$ $-$ $M -$

e no e e o d en on e e eq e M d en co n on of one d en on f nc on e e M e n e of n n o en On e o e nd e o d en on o ned y n co p c y ppo ed e e eq e on y ee c co n on c p e e con c on of e non nd d fo e ec on

II.5 A remark on computing in the wavelet bases

n y e no e once e e een c o en co p e e y de e ne e f nc on nd nd e efo e e e on n y n n e e n o e on n p ope y con c ed o e f nc on nd e ne e co p ed D e o e ec e de n on of e e e e n p on e pe fo ed e q d e o e nd e en f ey n o e q n e c ed nd A n e p e e co p e e o en of e n f nc on e e p e on fo e o en

$$\mathcal{M}_\infty^{\mathbf{m}} \;-\; \int \mathbf{m} \quad d \quad - \quad M -$$

n e of e e coe c en $\{\ {}_{\mathbf{k}}\}_{\mathbf{k}}^{\mathbf{k}\ L}$ y e fo nd n fo fo

$$- \;-\; = \prod_{\mathbf{j}}^{\infty} \quad -\mathbf{j}$$

e e

$$- \;=\; \sum_{\mathbf{k}}^{\mathbf{k}\ -} \ {}_{\mathbf{k}} e^{\mathbf{ik}}$$

e o en $\mathcal{M}^{m}_{\infty}$ e o ned n e c y n e de ed cc cy y ec e y ene n eq ence of ec o $\{\mathcal{M}^{m}_{r}\}^{m}_{m} \; M-$ fo r =

$$\mathcal{M}^{m}_{r+} = \sum_{j}^{j \; m} \quad ! \quad {}^{-j \, r}\mathcal{M}^{m-j}_{r}\mathcal{M}^{j}$$

n

$$\mathcal{M}^{m} = {}^{-m-\frac{1}{2}} \sum_{k}^{k \; k-} \quad k \quad m \quad = \quad M -$$

c ec o $\{\mathcal{M}^{m}_{r}\}^{m}_{m} \; M-$ ep e en M o en of e p od c n r e nd e on con e e p d y No ce e ne e co p ed e f nc on e f

e non nd d nd nd d fo

III.1 The Non-Standard Form

Le e n ope o

$$\mathbf{L} \quad \mathbf{R} \to \mathbf{L} \quad \mathbf{R}$$

e e ne y De n n p o ec on ope o on e p ce $\mathbf{V}_j$; $\in \mathbf{Z}$

$$P_j \quad \mathbf{L} \quad \mathbf{R} \to \mathbf{V}_j$$

$$P_j f = \sum_k \langle f \quad {}_{j;k} \rangle \quad {}_{j;k}$$

nd e p nd n n " e e op c e e e o n

$$= \sum_{j \in \mathbf{Z}} \quad j \quad j \quad j \quad P_j \quad P_j \quad j$$ e

e e

$$j = P_{j-} - P_j$$

e p o ec on ope o on e p ce $\mathbf{W}_j$ f e e e co e n en n e d of e e e

$$= \sum_{j = -\infty} \quad j \quad j \quad j \quad P_j \quad P_j \quad j \quad P_n \quad P_n$$

nd f e e ; = e ne e en

$$= \sum_j \quad j \quad j \quad j \quad P_j \quad P_j \quad j \quad P_n \quad P_n$$ 7

e e $\sim = P \quad P$ d e z on of e ope o on e ne e p n on e nd 7 deco po e ope o n o of con on f o d en e e

e non nd d fo ep e en on e 7 of e ope o c n of p e

$$= \{A_j \quad B_j \quad {}_j\}_{j \in \mathbf{Z}}$$

c n on e p ce $\mathbf{V}_j$ nd $\mathbf{W}_j$

$$A_j \quad \mathbf{W}_j \to \mathbf{W}_j$$

$$B_j \quad \mathbf{V}_j \to \mathbf{W}_j$$

$$\mathbf{W_j} \to \mathbf{V_j}$$

e e e ope o $\{A_j\ B_j\ \ j\}_{j\in Z}$ e de ned A_j := j j B_j := j P_j nd j := P_j j

e ope o $\{A_j\ B_j\ \ j\}_{j\in Z}$ d ec e de n on e e on

$$j := \begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix}$$

e e ope o j := P_j P_j

$$j \quad \mathbf{V_j} \to \mathbf{V_j}$$

nd e ope o ep e en ed y e × n pp n

$$\begin{pmatrix} A_{j+} & B_{j+} \\ j+ & j+ \end{pmatrix} \quad \mathbf{W_{j+}} \oplus \mathbf{V_{j+}} \to \mathbf{W_{j+}} \oplus \mathbf{V_{j+}}$$ e

f e e co e n en

$$:= \{\{A_j\ B_j\ \ j\}_{j\in Z\ j\le n}\ \ n\}$$

e e n := P_n P_n f e n e of e n e en := n n nd e ope o e o nzed oc of e ee. e nd

Le e e fo o n o e on

e ope o A_j de e e n e c on on e e on y nce e p ce $\mathbf{W_j}$ n n e e en of e d ec n

e ope o B_j j n nd de e e n e c on e een e e nd co e e ndeed e p ce $\mathbf{V_j}$ con n e p ce $\mathbf{V_{j'}}$ ' ee

e ope o j n e ed e on of e ope o j−

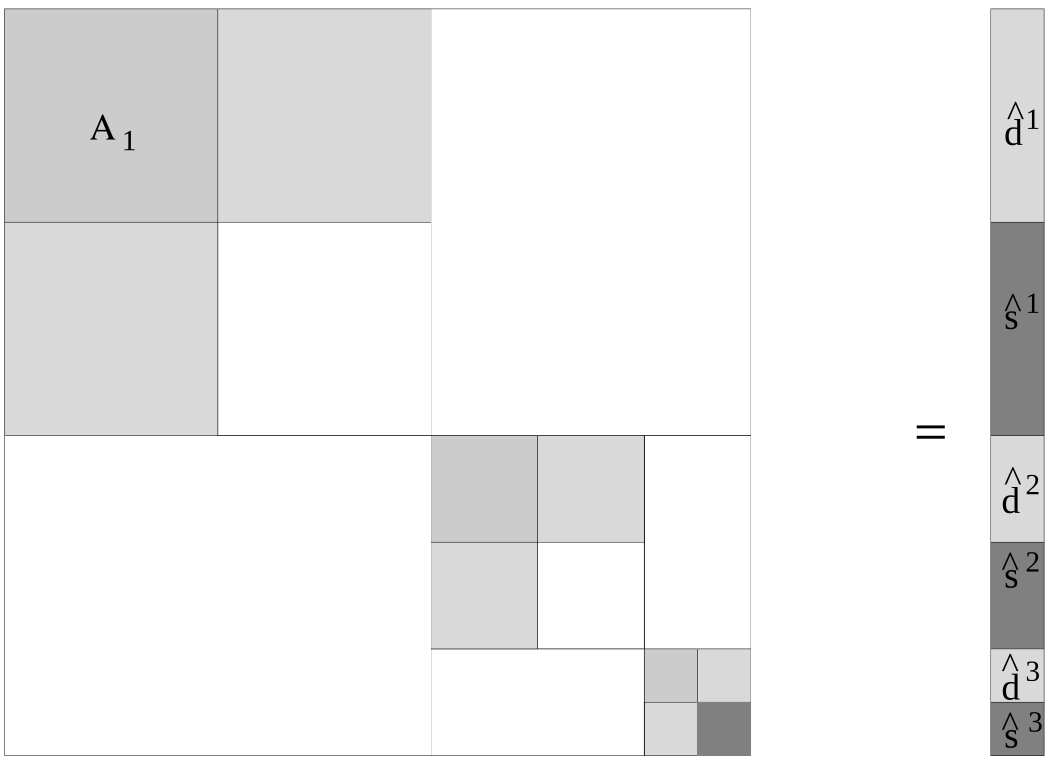
e ope o A_j B_j nd j e ep e en ed y e ce j j nd j

$$^{j}_{k;k'} := \int\int \quad y \quad j;k \quad j;k' \ y \ d\ dy$$

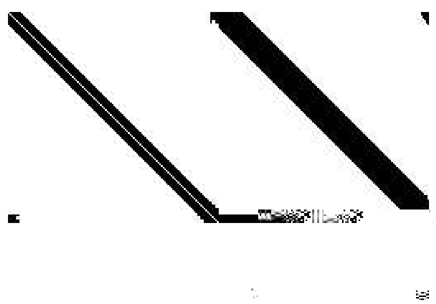
$$^{j}_{k;k'} := \int\int \quad y \quad j;k \quad j;k' \ y \ d\ dy \qquad 7$$

nd

$$^{j}_{k;k'} := \int\int \quad y \quad j;k \quad j;k' \ y \ d\ dy$$



A 1
=
d̂ 1
ŝ 1
d̂ 2
ŝ 2
d̂ 3
ŝ 3



e An e p e of n e non nd d fo ee p e

e ope o j ep e en ed y e j

$$ {}^{j}_{k;k'} = \int \int y \; {}_{j;k} \; {}_{j;k'} \; y \; d \; dy $$

en e of coe c en $_{k;k'}$ $' =$ $N -$ epe ed pp c on of e fo 7 p od ce

$$ {}^{j}_{i;l} = \sum_{k;m}^{L} {}_{k} \; {}_{m} \; {}^{j-}_{k+ \; i+ \; ;m+ \; l+} $$

III.2 The Standard Form

e nd d fo o ned y ep e en n

$$\mathbf{V}_{\mathbf{j}} = \bigoplus_{\mathbf{j}' > \mathbf{j}}^{\mathrm{M}} \mathbf{W}_{\mathbf{j}'}$$

nd con de n fo e c e e ope o $\{B_{\mathbf{j}}^{\mathbf{j}'}, \Gamma_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}'>\mathbf{j}}$

$$B_{\mathbf{j}}^{\mathbf{j}'} \; \mathbf{W}_{\mathbf{j}'} \rightarrow \mathbf{W}_{\mathbf{j}}$$

$$\Gamma_{\mathbf{j}}^{\mathbf{j}'} \; \mathbf{W}_{\mathbf{j}} \rightarrow \mathbf{W}_{\mathbf{j}'}$$ 7

f e e e co e en n en n e d of e e

$$\mathbf{V}_{\mathbf{j}} = \mathbf{V}_{\mathbf{n}} \bigoplus_{\mathbf{j}' \; \mathbf{j}+}^{\mathbf{j}' \mathrm{M} \mathbf{n}} \mathbf{W}_{\mathbf{j}'}$$

n c e e ope o $\{B_{\mathbf{j}}^{\mathbf{j}'}, \Gamma_{\mathbf{j}}^{\mathbf{j}'}\}$ fo $j' =$ n e e e n nd 7 nd n dd on fo e c e e e ope o $\{B_{\mathbf{j}}^{\mathbf{n}+}\}$ nd $\{\Gamma_{\mathbf{j}}^{\mathbf{n}+}\}$

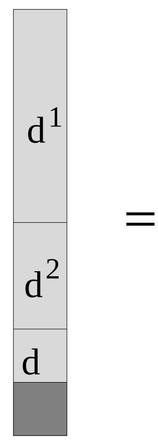
$$B_{\mathbf{j}}^{\mathbf{n}+} \; \mathbf{V}_{\mathbf{n}} \rightarrow \mathbf{W}_{\mathbf{j}}$$

$$\Gamma_{\mathbf{j}}^{\mathbf{n}+} \; \mathbf{W}_{\mathbf{j}} \rightarrow \mathbf{V}_{\mathbf{n}}$$

n no on $\Gamma_{\mathbf{n}}^{\mathbf{n}+} = \Gamma_{\mathbf{n}}$ nd $B_{\mathbf{n}}^{\mathbf{n}+} = B_{\mathbf{n}}$ f e n e of e n e nd $\mathbf{V}$ n e d en on en e nd d fo ep e en on of $= P \; P$

$$= \{A_{\mathbf{j}} \; \{B_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}' \; \mathbf{j}+}^{\mathbf{j}' \; \mathbf{n}} \; \{\Gamma_{\mathbf{j}}^{\mathbf{j}'}\}_{\mathbf{j}' \; \mathbf{j}+}^{\mathbf{j}' \; \mathbf{n}} \; B_{\mathbf{j}}^{\mathbf{n}+} \; \Gamma_{\mathbf{j}}^{\mathbf{n}+} \; \mathbf{n}\}_{\mathbf{j} \; ;:::;\mathbf{n}}$$

e ope o e o n zed oc of e ee e nd e e f e ope o C de on Zy nd o p e do o



d1
d2
d
=

e co p e on of ope o

e co p e on of ope o o n o e o d e con c on of e p e ep e en on n o ono e d ec y ec e peed of co p on o e e co p e on of d of e fo e p e c e ed y e od o e n nd n p e ep e en on n o e y e deq e fo o e pp c on e co p e on of ope o c fo ep e en on n n o de o e ec e y co p e n e p e fo e nd d nd non nd d fo of ope o n e e e y e e ed co p e on e e e fo de c of nd non nd d fo of ope the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| _{i;l}^{j}| \quad | _{i;l}^{j}| \quad | _{i;l}^{j}| \le \frac{C_M}{|i - l|^{M+}}$$

for all $|i - l| \ge M$.

con de on pp y n e c e of p do d en ope o Le e p do d en ope o y o de ned y e fo

$$f \quad - \; e^{ix} \int f \; d - \int \; y \; f \; y \; dy$$

e e e d on e ne of

Proposition IV.2 **If the wavelet basis has M vanishing moments, then for any pseudo-di erential operator with symbol of and * of * satisfying the standard conditions**

$$| \quad _x \quad | \le C \, ; \qquad | \quad - \; +$$

$$| \quad _x \; ^* \quad | \le C \, ; \qquad | \quad - \; +$$

the matrices ^{j}, ^{j}, ^{j} (3.16) - (3.18) of the non-standard form satisfy the estimate

$$| _{i;l}^{j}| \quad | _{i;l}^{j}| \quad | _{i;l}^{j}| \le \frac{^j \, C_M}{|i - l|^{\;M+}}$$

for all integer i, l.

f e pp o e e ope o ^{N} y e ope o $^{N;B}$ o ned f o ^{N} y e n o ze o coe c en of ce $^j_{i;l}$ $^j_{i;l}$ nd $^j_{i;l}$ o de of nd of d $B \ge M$ o nd e d on en e y o ee

$$|| \; ^{N;B} - \; ^N|| \le \frac{C}{B^M} \; \text{o} \; N$$

e e C con n de e ned y e e ne n o n e c pp c on e cc cy of c c on ed nd e p e e of e o n o c e nd d B nd o de M e o e c o en n c nne e de ed p ec on of c c on c e ed f M ed en B o e c

$$|| \; ^{N;B} - \; ^N|| \le \frac{C}{B^M} \; \text{o} \; N \le$$

y — * y

e ef c y n y n e f nc on e e nd e nd e efo e e ope o L nd L^* po e o dec de f C de on Zy nd ope o o nded

Theorem IV.1 (G. David, J.L. Journe) **Suppose that the operator (3.1) satis es the conditions (4.5), (4.6), and (4.16). Then a necessary and su cient condition for to be bounded on** L **is that in (4.24) and** y **in (4.25) belong to dyadic** BMO**, i.e. satisfy condition**

$$\sup_J \frac{1}{|J|} \int_J | \quad J \quad | \, d \le C$$

where is a dyadic interval and

$$J \quad \frac{1}{|J|} \int_J \quad d$$

p n e ope o n o e of ee e e nd e n e p e y e d o e e e e e no e e f nc on nd * e e y co p ed n e p oce of con c n e non nd d fo nd nd y e ed o p o de ef e e of e no of e ope o

e d e en ope o n e e e

V.1 The operator d=dx in wavelet bases

e non nd d fo of e e co on y ed ope o y e co p ed e p c y e con c e non nd d fo of e ope o d d e e e en of A_j B_j nd r_{il}^{j} of [illegible]

e e n e e oco e on coe c en of e e $\{ k\}_k^{L-}$

$$n \quad L-X-n \quad i \quad i+n \quad n \quad L-$$

e y o ee e oco e on coe c en n e en nd ce e ze o

$$k \quad L-$$

y e e fyed y n o co p e | nd |

$$| \quad L X \quad n \quad n \quad \cos n$$

$$| \quad L= X \quad k- \quad \cos \quad - \quad L= X- \quad k \quad \cos$$

e e n e en n Co n n nd o fy e o n

$$L=X- \quad k \quad k \quad \cos$$

nd ence nd

e e en o en of e coe c en k- f o n n e y

$$k \; k= X \quad k- \quad - \quad m \quad \text{fo} \quad \leq \quad \leq M-$$ e

nce

$$- \quad m \quad | \quad \text{fo} \quad \leq \quad \leq M-$$

c fo o f o e Co o d en f e 7 e d e y fy o 7 7 e f d cof e n e e e 7

$$r_i := \sum_{k}^{L} \sum_{m}^{k-1} k \; k-m \; r \; i+m$$

C n n e o de of on n nd n e f c $\sum_{k}^{L-1}$ k := e e

$$r_l := r_l \sum_{n}^{L} n \; r_{l-n} \; r_{l+n} \quad \in \mathbf{Z}$$

e e n e en n n e o n e f o n o de o o n e e fo o n e on

$$\sum_{l=-\infty}^{+\infty} m \quad - \quad := m \sum_{l}^{m} - l \qquad M_l \; m-l$$

e e

$$M_l := \int_{-\infty}^{+\infty} l \, d \quad :=$$

e e o en of e f nc on e on fo o p y on n o e n fo nd n Le n z e n nd := e o n f $M \geq$ en

$$| \; | \; | \leq C \quad | \; |^{- -}$$

e e nd ence e n e n e y con e en on fo o f o Le of e e o n

$$| \; | \leq C \quad | \; |^{-M + \log_2 B}$$

e e

$$B := \sup_{\in \mathbf{R}} | \; e^{i} \; |$$

D e o e cond on e e o $B := M - -$ e e e ence of on of e e of eq on e nd fo o f o e e ence of e n e n nce e n f nc on co p c ppo e e e on y 7 7 d ee e d 7 d e e e

$$r \; = \sum_{l} r_l e^{il}$$

$$r_{even} \; = \sum_{l} r \; _l e^{il}$$ 7

nd

$$r_{odd} \; = \sum_{l} r \; _{l+} \; e^{i \; l+ \;) \; =}$$

No c n

$$r_{even} \; = r \; \; r$$

nd

$$r_{odd} \; = r \; - r$$ e

nd n e o n f o

$$r \; = r \; \; r \; \; \left[\; \; | - \; r \; - r \right]$$ e

n y n e e

$$r \; = \; \; | r \; \; | r$$ e

e n = n e e o n r = r nd e
n q ene of e o on of e nd fo o f o e n q ene of
e ep e en on of d d en e on r_l of e nd e con de e
ope o $_j$ de ned y e coe c en on e p ce $\mathbf{V}_j$ nd pp y o c en y
oo f nc on f nce r^j_l = $^{-j} r_l$ e e e

$$_j f \; = \sum_{k \in Z} \; ^{-j} \left(\sum_{l} r_l f_{j;k-l} \; \right) _{j;k}$$ e

e e

$$f_{j;k-l} = \; ^{-j=} \int_{-\infty}^{+\infty} f \; \; ^{-j} \; - \; \; d$$ ee

e n ee

$$f_{j;k-l} =$$

d 7 7

e e — j nd | − | ≤ j n e n e nd n e
nd e o n

$$ j f — \sum_{k\in Z} \int_{-\infty}^{+\infty} f' \quad j;k \quad d \quad j;k $$

$$ j \sum_{k\in Z} \sum_{l} - r_l \int_{-\infty}^{+\infty} f'' \quad j;k \quad d \quad j;k \qquad ! $$

c e → −∞ ope o j nd d d co nc de on oo f nc on n nd d en e y o p o e −∞ —d d nd ence e o on o e nd n q e e e on 7 fo o no f o

Remark 2 e no e e p e on nd fo l nd l l — − −l y e p ed y c n n e o de of on n nd nd n od c n e co e on coe c en $\sum_{i}^{L-\,-n}$ i i+n $\sum_{i}^{L-\,-n}$ i i+n nd $\sum_{i}^{L-\,-n}$ i i+n e e p e on fo l e pec y p e l —e r l − r l

Examples. o e e p e e D ec e e e con c ed n e co p e e coe c en k— — M e e M en e of n n o en nd L — M n e on e of

$$ | — - \frac{M -}{M - \qquad M-} \int n^{M-} d $$

e nd y co p n R n M− d

$$ | —- \; -C_M \sum_{m}^{M} \frac{- \; m- \; co \quad - }{M \qquad M \quad - \qquad -} $$

e e

$$ C_M — \left[\frac{M -}{M - \quad e^{M-}} \right] \# $$

y co p n e nd e e

$$ m- \; — \frac{- \; m- \; C_M}{M \qquad M \quad - \qquad -} \qquad e e \; — \qquad M $$

e no e y e of en o on of ne y e on co e c en m− n e on y con c on e coe c en r l e on n e e coe c en r l e e e fo D ec e e e ed n e of n n o en M e e e e e e e fo en

j e d e d en

j ene e
of ep d o yno e

o n eq on of opo on e p e en e e fo D ec e e e
M e

1 M

nd

r r

e coe c en – – of e p e c n e fo nd n ny oo on n e c n y c o ce of coe c en fo n e c d en on

---

2 M

nd

r r r r_4

---

3 M

nd

r r r

r_4 r r

---

4 M

nd

r r r

r_4 r

nd

Coe c en fo M = nd M = c n e co p ed e co e pond n o p f o e fo o n e e o

Iterative algorithm for computing the coe cients r_l.

A y of o n eq on e nd e y o e n e e o e r = nd r = − nd e e n e o eco p e r_l e y o e fy n e nd e ed d e o e c o ce of n z on e fo o n fo D ec e e e M = co p ed n o d p y e coe c en $\{r_l\}$ e no e r_{-l} = $-r_l$ nd r =

V.2 The operators d^n=dx^n in the wavelet bases

o e ope o d d e non nd d fo of e ope o d^n d^n co p e e y de e ned y ep e en on on e p ce **V** e y e coe c en

$$r_l^{(n)} = \int_{-\infty}^{+\infty} \; - \; \frac{d^n}{d^n} \; d \quad \in \mathbf{Z}$$

o e n e y

$$r_l^{(n)} = \int_{-\infty}^{+\infty} - \xi^n |\;\xi| \; e^{-il} \, d\xi$$

f e ne n o e ee e e o

	i	Coe cients			i	Coe cients
		I				I
M = 5	1	-0.82590601185015		M = 8	1	-0.88344604609097
	2	0.22882018706694			2	0.30325935147672
	3	-5.3352571932672E-				

Proposition V.2 **1. If the integrals in (5.52) or (5.53) exist, then the coe cients $r_l^{n)}$, $\in$ Z satisfy the following system of linear algebraic equations**

$$r_l^{n)} = {}^{n} \left[r_l - \sum_{k}^{L=} {}_{k-} \; r_{l-k+}^{n)} \quad r_{l+k-}^{n)} \right] \qquad 4$$

and

$$\sum_l {}^{n} r_l^{n)} = - \; {}^{n} n$$

where ${}_{k-}$ **are given in (5.19).**

2. Let $M \geq n$, where M is the number of vanishing moments in (2.16). If the integrals in (5.52) or (5.53) exist, then the equations (5.54) and (5.55) have a unique solution with a nite number of non-zero coe cients $r_l^{n)}$, namely, $r_l^{n)} \neq$ for $-L \leq \leq L -$. Also, for even n

$$r_l^{n)} = r_{-l}^{n)}$$

$$\sum_l {}^{n} r_l^{n)} = \quad n = \quad n \; - \qquad 7$$

and

$$\sum_l r_l^{n)} =$$

and for odd n

$$r_l^{n)} = -r_{-l}^{n)}$$

$$\sum_l {}^{n-} r_l^{n)} = \quad n = \qquad \leq \leq L$$

$$A \quad e \; M$$

e no e on e e e L = e e e e o n n o en M = do no e e o de e ponen ee e ep e en on of e d de e e on y f e n e of n n o en M =

e eq on fo co p n e coe c en $r_l^{(n)}$ y e e ed n e en e p o e Le de e e eq on co e pond n o e fo d^n d^n d ec y f o e e e

$$r_l^{(n)} = \int \sum_{k\in Z} | \quad |^n \quad {}^n e^{-il} \, d$$

e efo e

$$r = \sum_{k\in Z} | \quad |^n \quad {}^n$$

e e

$$r = \sum_l r_l^{(n)} e^{il}$$

n e e on

e

n o e nd de of nd n o e e en nd odd nd ce n ep e y e e

$$r = {}^n \quad | r \quad | r$$

Le con de e ope o M on pe od c f nc on d f n f d

$$M f = \quad | f \quad | \quad |$$

n e e e e n e c e dence n f c of n e e nce ep
e en one of e d n e of co p n n e e e e
e e c e

N	[illegible]	[illegible]p
64	0.14545E+04	0.10792E+02
128	0.58181E+04	0.11511E+02
256	0.23272E+05	0.12091E+02
512	0.93089E+05	

e con o on ope o n e e e

n ec on e con de e co p on of e non nd d fo of con o on ope o . o con o on ope e q d e fo fo ep e en n e e ne on **V** e of e p e fo d e

nd e den y fo o f o
o d

nce e o en of e f nc on n eq on e e d o
po n q d e fo fo co p n e ep e en on of con o on
e ne e fo co p c y ppo ed e e fo o
e e nne fo e pec c o ce of e e e de c eqn
e e e f ed o en of e f nc on n e ef p pe fo
e de

e e e n od ce d en pp o c c con n of
ne e c eq on ec o y p o c cond on pe
c y p e f e y o of e ope o o o eneo of e
c e ope o co p e e y de ned y ep e en on on **V**

Le con de o e p e of c ope o e nd
ope o of f c on d en on o n d en on

VI.1 The Hilbert Transform

e pp y o e od o e co p on of e non nd d fo e n
fo

$$\mathcal{H}f \quad y \quad = \mathrm{p} \int_{-\infty}^{\infty} \frac{f}{\;-\;} d$$

e e p deno e p nc p e

e ep e en on of $\mathcal{H}$ on **V** de ned y e coe c en

$$r_{\mathbf{l}} = \int_{-\infty}^{\infty} \; - \; \mathcal{H} \; d \; \in \mathbf{Z}$$

c n n co p e e y de ne o e coe c en of e non nd d

$\mathcal{H} = \{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j}\in\mathbf{Z}}$ $A_{\mathbf{j}} = A$ $B_{\mathbf{j}} = B$ nd $\mathbf{j} =$ e e e

		Coe cients		Coe cients
M = 6	1	-0.588303698	9	-0.035367761
	2	-0.077576414	10	-0.031830988
	3	-0.128743695	11	-0.028937262
	4	-0.075063628	12	-0.026525823
	5	-0.064168018	13	-0.024485376
	6	-0.053041366	14	-0.022736420
	7	-0.045470650	15	-0.021220659
	8	-0.039788641	16	-0.019894368

e e e coe c en r_l of e n fo fo D ec e e e n n o en

e coe c en $r_l \in$ **Z** n fy e fo o n e of ne e c eq on

$$r_l \ldots r_l - \sum_{k} \ldots r_{l-k+} \ldots r_{l+k-}$$

e e e coe c en k− e en n n e nd 7 e o n e y p o c of r_l fo e

$$r_l \ldots O \frac{}{M}$$

By e n n e of

$$r_l \ldots \int^{\infty} |\ldots| \ldots n \ldots d\ldots$$

e o n $r_l \ldots r_{-l}$ nd r e no e e coe c en r c nno e de e ned f o eq on e nd o n e e y p o c cond on e co p e e coe c en $r_l \neq$ ny p e ed cc cy

Example.

e co p e e coe c en r_l of e n fo fo D ec e e e n n o en cc cy − e coe c en fo e o ned n y p o c e no e $r_{-l} \ldots r_l$ nd r

VI.2 The fractional derivatives

e e fo o n de n on of f c on de e

$$ {}_{x} f \;=\; \int_{-\infty}^{+\infty} \frac{-\,y\ \ -\ -\ +}{\ -} f\ y\ dy \qquad 7 $$

e e e con de $\neq$ f en 7 de ne f c on n de e
e ep e en on of $_{x}$ on **V** de e ned y e coe c en

$$ r_{l} = \int_{-\infty}^{+\infty} - \quad x \quad d \qquad \in \mathbf{Z} $$

p o ded n e e
e non nd d fo $_{x} = \{A_{j}\ B_{j}\ \ _{j}\}_{j\in Z}$ co p ed $A_{j} = -\ {}^{j}A$ $B_{j} = -\ {}^{j}B$ nd $_{j} = -\ {}^{j}$ e e e e en $_{i-l}$ $_{i-l}$ nd $_{i-l}$ of A B nd
e o ned f o e coe c en r_{l}

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} \quad {}_{k}\ {}_{k'}\ r\ {}_{i+k-k'} $$

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} \quad {}_{k}\ {}_{k'}\ r\ {}_{i+k-k'} $$

nd

$$ {}_{i} = \sum_{k}^{L} \sum_{k'}^{L} \quad {}_{k}\ {}_{k'}\ r\ {}_{i+k-k'} $$

e y o e fy e coe c en r_{l} fy e fo o n y e of ne e c eq on

$$ r_{l} = \left[r_{\ l} - \sum_{k}^{L} \quad {}_{k-}\ \ r\ {}_{l-\ k+} \quad r\ {}_{l+\ k-} \right] $$

e e e coe c en $_{k-}$ e en n n e nd 7 e o n
e y p o c of r_{l} fo e

$$ r_{l} = \frac{\ }{\ -}\,\frac{\ }{+} \quad O\,\frac{\ }{+\ +\ M} \qquad \text{fo} $$

$$ r_{l} = \qquad \text{fo} $$

Example.

		Coe cients		Coe cients
	l	ı	l	ı
M = 6	-7	-2.82831017E-06	4	-2.77955293E-02
	-6	-1.68623867E-06	5	-2.61324170E-02
	-5	4.45847796E-04	6	

M p c on of ope o n e e e

VII.1 Multiplication of matrices in the standard form

e p c on of ce of C de on Zy nd nd p e do d en ope o n e nd d fo eq e o O N o N ope on n dd on po e o con o e d of e n e nd y e n o ze o e en e n e p od c e o e o d of nd e efo e

$$|| \quad - \quad || \leq \qquad (7\ldots)$$

e nd de of 7 do n ed y , o e p e f e co p e [4] en e o e one n c n d

VII.2 Multiplication of matrices in the non-standard form

e no o ne n o fo e p c on of e ope o n e non nd d fo ne o e e n e y deco p e e e n e p oce of p c on Le nd e o ope o

$$\mathbf{L} \quad \mathbf{R} \to \mathbf{L} \quad \mathbf{R} \qquad (7.7)$$

en e non nd d fo of nd $\{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j} \in \mathbf{Z}}$ nd $\{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j} \in \mathbf{Z}}$ e co p e e non nd d fo $\{A_{\mathbf{j}} \; B_{\mathbf{j}} \; \mathbf{j}\}_{\mathbf{j} \in \mathbf{Z}}$ of —

e ec e ope o of

n y e e e 7 of o e

7

e e

$$\sum_{j}^{n} \big[A_j A_j \quad B_j \quad B_j \; j \quad A_j B_j \quad j \quad A_j \big] \qquad 7$$

nd

$$n \; n \sum_{j}^{n} P_j \, j B_j P_j \qquad 7$$

e ope o n e 7 e c n on fo o n e p ce

$$A_j A_j \quad B_j \quad \mathbf{W}_j \to \mathbf{W}_j \qquad 7 \quad e$$

$$B_j \; j \quad A_j B_j \quad \mathbf{V}_j \to \mathbf{W}_j \qquad 7$$

$$j \quad j A_j \quad \mathbf{W}_j \to \mathbf{V}_j \qquad 7$$

nd e ope o n e 7

$$j B_j \quad \mathbf{V}_j \to \mathbf{V}_j \qquad 7 \; 7$$

e e n 7 d d e f d 7 7 f

of ope on ed e c e e o o e p e e e o n e of
ope on ep p opo on o N
e e n ope o A_j B_j j n tot49 -410.315ac5 0 Td (36su8712.7097 0552

e e o n e e e

e f p c on o of ec on e econd fe o n e of e e o

VIII.1 An iterative algorithm for computing the generalized inverse

n o de o

p oced e nd e e on n e e e e n
fo e co p on e e pe fo ed on n p c o on nd e ed
o ne f o L N AC fo co p n e n e deco po on o e e
ed e fo o n f n

$$A_{ij} = \begin{cases} \frac{1}{i-j} & \\ & \end{cases}$$

e e N e cc cy e o d e o $^{-4}$ e en e of X_k e o
$^{-4}$ e e e c y e o ed f e e c e on

Size $N \times N$	SVD	FWT Generalized Inverse	L_2-Error
128×128	20.27 sec.	25.89 sec.	$3\,1 \cdot 10^{-4}$
256×256	144.43 sec.	77.98 sec.	$3\,42 \cdot 10^{-4}$
512×512	1,155 sec. (est.)	242.84 sec.	$6\,0 \cdot 10^{-4}$
1024×1024	9,244 sec. (est.)	657.09 sec.	$7\,7 \cdot 10^{-4}$
...	...	...	...
$2^{15} \times 2^{15}$	9.6 years (est.)	1 day (est.)	

Le de e e e e e o nd c n n e c f nc on
c c ope o c n e p e en ed e c en y e fo p e do d en op
e o N e c e nd e e pe fo nce of e o e epo ed
ep e y

VIII.2 An iterative algorithm for computing the projection operator on the null space.

Le con de e fo o n e on

$$X_{k+} = X_k - X_k$$

$$X = A^* A$$

e e A^* e d o n nd c o en

en $-X_k$ con e e o P_{null} c n e o n e e d ec y o y co n n n n n ep e en on fo $P_{null} := -A^*(AA^*)^{\dagger}A$ e e on o co p e e ene zed n e $(AA^*)^{\dagger}$ e f p c on o e e e on e f fo de c of ope o e e co p e y e o fo e ene zed n e e po n d ence o e e e doe no eq e co p e y of e n e ope o on y of e po e of e ope o

VIII.3 An iterative algorithm for computing a square root of an operator.

Le de e n e on o con c o A nd A^{-} e e A fo p c y f d o n nd non ne e de n e ope o e con de e fo o n e on

$$Y_{l+} := Y_l - Y_l X_l Y_l$$

$$X_{l+} := - X_l \quad Y_l A$$

$$Y := - A$$

$$X := - A \tag{7}$$

e e c o en o e e e en e of $-A$ e n $\sqrt{\ }$ e eq ence X_l con e e o A nd Y_l o A^{-} By n $A := V^* D V$ e e D d on nd V n y e y o e fy o X_l nd Y_l c n e en $X_l := V^* P_l V$ nd $Y_l := V^* \ _l V$ e e P_l nd $_l$ e d on nd

$$_{l+} := \ _l - \ _l P_l \ _l$$

P e e

r n f f f d e y A f d f d

VIII.4 Fast algorithms for computing the exponential, sine and cosine of a matrix

e e ponen of o n ope o e ne nd co ne f nc on e on e o e con de ed n ny c c of ope o A n e c e of e ene zed n e e

X Co p n $F(u)$ n e e e e

n ec on e de e f d p e o fo co p n e e n n n e y d en e f nc on nd ep e en ed n e e An po n e p e O n y c e ene ze eo e of M Bony on e p op on of n e of on of non ne eq on O n e c pp o c o e e no e e e pec de n e of pp c on of o

IX.1 The algorithm for evaluating u^2

e n o o co p e Le p o ec $\in$ **L R** on p ce $\mathbf{V_j}$ $\in$ **Z**

$$_j = P_j \quad _j \in \mathbf{V_j}$$

n o de o deco p e e e e e e cop c e e

$$- \quad _n = \sum_j^n \quad P_{j-} \quad - P_j \quad = \sum_j^n P_{j-} \quad P_j \quad P_{j-} \quad - P_j$$

n $P_{j-} = P_j$ $_j$ e o n

$$- \quad _n = \sum_j^n \quad P_j \quad _j \quad _j$$

o

$$= \sum_j^n P_j \quad _j \quad \sum_j^n \quad _j \quad _j \quad _n$$

n e e e no n e c on e een d en e e nd
o e n e c p po e need fo o e n e n e of e o c e

Befo e p oceed n f e e con de n e p e of e n

e e e fo o n e p c e on

$$
\begin{aligned}
&{}^{j}_{k} \quad - \quad -j= \quad {}^{j}_{k} \\
&{}^{j}_{k} \quad - \quad -j= \quad {}^{j}_{k} \\
&{}^{j}_{k} \quad {}^{j}_{k} \quad - \quad -j= \quad {}^{j}_{k}
\end{aligned}
\qquad 7
$$

A o e p od c on e e e e zeo

p nd n e p c y n o

$$- \sum^{n}_{j} \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \qquad \sum_{k \in Z} {}^{n}_{k} \; {}^{n}_{k}$$

nd n 7 e o n f o e

$$- \sum^{n}_{j} \; -j= \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \; {}^{j}_{k} \qquad \sum^{n}_{j} \; -j= \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \qquad -n= \sum_{k \in Z} {}^{n}_{k} \; {}^{n}_{k}$$

On deno n

$$
\begin{aligned}
&d^{j}_{k} \quad - \quad -j= + \; d^{j}_{k} \; {}^{j}_{k} \\
&{}^{j}_{k} \quad - \quad -j= \; d^{j}_{k} \\
&{}^{n}_{k} \quad - \quad -n= \; {}^{n}_{k}
\end{aligned}
$$

e e e

$$- \sum^{n}_{j} \sum_{k \in Z} d^{j}_{k} \; {}^{j}_{k} \qquad \sum^{n}_{j} \sum_{k \in Z} {}^{j}_{k} \; {}^{j}_{k} \qquad \sum_{k \in Z} {}^{n}_{k} \; {}^{n}_{k}$$

e no e **if the coe cient d^{j}_{k} is zero then there is no need to keep the corresponding average** ${}^{j}_{k}$ n o e o d e need o eep e e on y ne e n e e e e e e e coe c en d^{j}_{k} o p od c ${}^{j}_{k} d^{j}_{k}$ e n c n fo en cc cy

of coe c en c need o e o ed y e ed ced f e y o e n fo e p e

$$M^{\mathbf{j};\mathbf{j}'}_{\mathbf{w}\mathbf{w}\mathbf{w}} \quad ' \quad - \quad {}^{-\mathbf{j}'=} \int_{-\infty}^{+\infty} \quad \mathbf{j}-\mathbf{j}' \quad {}^{\mathbf{j}-\mathbf{j}'}_{\mathbf{k}-\mathbf{k}'} \quad {}^{j-j'}\mathbf{k}-\mathbf{l} \quad d$$

o

$$M^{\mathbf{j};\mathbf{j}'}_{\mathbf{w}\mathbf{w}\mathbf{w}} \quad ' \quad - \quad {}^{-\mathbf{j}'=} M^{\mathbf{j}-\mathbf{j}'}_{\mathbf{w}\mathbf{w}\mathbf{w}} \quad - \quad ' \quad \mathbf{j}-\mathbf{j}' \quad -$$

o e e e o n c n ed c on n e n e of coe c en con e q ence of e f c e coe c en n dec y e d nce $r \; - \; - \; '$ e een e e f en e of n c n coe c en d_k^j p opo on o e n e of e o N e e n e of ope on eq ed o e e e pp n

o c e nece y o o e on y o e e $_{k'}^{j}$ c co ne e n c n coe c en d_k^j o p od ce non ze o con on e efo e c en o o e on y o $_{k'}^{j}$ fo c e e e e coe c en d_k^j c $| - '| \leq$ nd e p od c $_{k'}^{j} d_k^j$ o e e e o d of cc cy e n e need o o e e e on y n e ne o ood of n e

en e of ope on fo e p nd n of e econd e n n o e e e p opo on o e n e of n c n en e nd e e e co p e e y o fo [redacted]

Remark. e o fo e on n e e e o o e e e p od c of o f nc on nce $\frac{}{4}$

IX.2 The algorithm for evaluating F(u)

Le e n n n e y d en e f nc on n o de o deco p e e e e n e e op c e e

$$- \quad {}_{n} \quad \sum_{j}^{n} \quad P_{j} - \quad - \quad P_{j} \qquad (7)$$

p nd n e f nc on n e y o e e e po n y e P d 7 7 e de p

e no e e e no e e econd de e of n e e e no e en
de e n n nd con de n e e nde of e e e n
n e o e e o n e e of M Bony e e o e o e e e
y oo e n n Bony e nce e e nde f c o j n e d
of j e y eep o e e o e e e nde y oo
no c e po n f e e n d n e n co p n
o epe ed pp c on of e o fo c en o co p e o
f nc on o e e e e e e n y c d n e n con de n n
p c y n eo e c c e z n e

efe ence

. , B A pe p e ep e en on of oo ne ope o D e Y e
n e y

. , C e L een d nd o n A f d p e po e o fo p c e on SIAM Journal of Scienti c and Statistical Computing e Y e n e y ec n c epo YALeB DOo e

e A Co en D ec e nd C .

M e e of f eq ency c nne deco po on of e nd e e ode ec n c epo e Co n n e of M e c c ence Ne Yo n e y

Y Meye Le c c en q e e onde e e e e o en q d e C MAD n e e D p ne

Y Meye nc pe d nce de e e enne e e e d ope e n