Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
January, 2020

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless

(c) Use Newton's method on $f(x) = e^x - \sin(x)$. Use arguments similar to those in (a) to argue that $f'(x) \neq 0$ for $x \in [-\pi/2, \pi/2]$.

2. **Linear Alegbra.**

(a) Let A be a real $n \times n$ matrix with distinct eigenvalues such that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n| \geq 0$$

with corresponding eigenvectors $\{v_j$

(b) We need to rewrite y_{n+1} so that it does not involve x_{n+1}. We do this by simply plugging in the definition of x_{n+1} to find $y_{n+1} = 2x_n + y_n$.
Then the linear system iteration is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

(c) We know that the power iteration converges to the eigenve for some $\eta \in [a, b]$ by the mean value theorem. (See for example Chapter 1 Thm 1.3 of Atkinson Numerical Analysis text.) The trapezoidal rule is given by $I_1(f) = \frac{f(a)+f(b)}{2}(b-a)$ and the error term is $E_1(f) = -f''(\eta)\frac{(b-a)^3}{12}$.

(b) $I_n(f) = h\left[\frac{f(x_0)}{2} + f(x_1) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right]$

(c)

$$E_n(f) \leq$$

So we need to chose a and b so that l_1 satisfies the first two conditions. After some algebra you find a = ^{2}

Solution (b)

Plug in the right hand side to find

$$y_{n+1} = y_n - \alpha h\lambda y_n + \beta h(-\lambda(y_n - \lambda\gamma h y_n)) = (1 - (\alpha + \beta)h\lambda + \beta\gamma h^2\lambda^2)y_n.$$

We know that for second order we require $\alpha + \beta = 1$ and $2\beta\gamma = 1$ so that in order for the sequence to be bounded

$$|1 - z + \frac{z^2}{2}| \le 1, \quad z = h\lambda.$$

Thus

$$-1 \le 1 - z + \frac{z^2}{2} \le 1.$$

or

$$\frac{z^2}{2} - z \le 0 \ \square\ z \ge 0, \quad z \le 2.$$

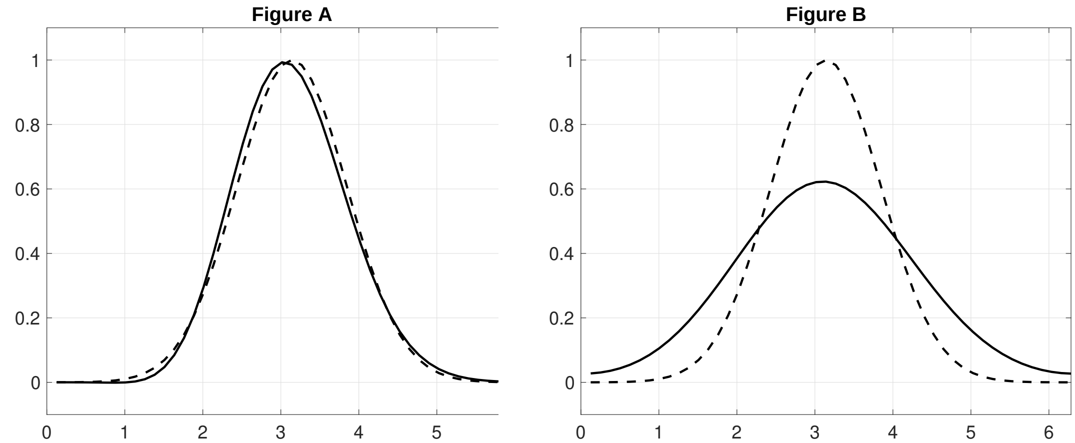
That is $h \le \frac{2}{\lambda}$.

To find the error estimate note that $y(t) = e^{-\lambda t}$ so that

$$y(t_n) - y_n = e^{-\lambda t_n} - (1 - h\lambda + \frac{h^2\lambda^2}{2})^n = (e^{-\lambda h})^n - (1 - h\lambda + \frac{h^2\lambda^2}{2})^n.$$

Recall that

[illegible]



Solid lines represents the numerical solution and dashed lines the exact solution.

(b) Let D denote one of the difference operators above. Then if we discretize in time using the trapezoidal rule we have (the superscript now denotes the time index)

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + D\left(\frac{v_j^{n+1} + v_j^n}{2}\right) = 0.$$

Show that with this timestepping the spatial discretization corresponding to "Figure A" satisfies $\|v^{n+1}\|_h^2 = \|v^n\|_h^2$ while the discretiztion corresponding to "Figure B" satisfies $\|v^{n+1}\|_h^2 \leq \|v^n\|_h^2$. Hint: First find α_+ and / or α_- such that $D_\pm v_j = D_0 v_j + \alpha_\pm D_+ D_- v_j$.

Solution (a):

The continuous problem can be treated by Fourier series. Assume that the expansion of the initial data is

$$u(x,0) =$$

Solution (b):
Multiply by $v_i^{n+1} + v_i^n$ and sum to find

$$\|v^{n+1}\|_h^2 - \|v^n\|_h^2 + \frac{\Delta t}{2}(v^{n+1} + v^n, D(v^{n+1} + v^n))_h = 0.$$

First note that for any periodic grid functions r, s we have $(r, D_0 s) = -(D_0 r, s)$ (just write out the expressions term by term and use the boundary conditions) so that

$$(v^{n+1} + v^n, D_0(v^{n+1} + v^n))_h = 0,$$

and the first part follows.

Second, as indicated by the hint, we have the identity

$$D_- v_j = D_0 v_j - \frac{h}{2} D_+ D_- v_j.$$

The second part then follows by noting that $(r, D_+ s) = -(D_- r, s)$ so that for scheme (1) we have

$$\|v^{n+1}\|_h^2 - \|v^n\|_h^2 + \frac{\Delta t h}{4}(D_-(v^{n+1} + v^n), D_-(v^{n+1} + v^n))_h = 0.$$

The

$$\|v^{n+1}\|_h^2 = \|v^n\|_h^2 - \Delta th$$