Applied Analysis Preliminary Exam, SOLUTIONS

10.00am–1.00pm, August 21, 2018

Problem 1 Solution:

(a) Fix any x, y R^n. We want to show that $(g - h) \leq 0$. Note that

$$(g - h)(0) = 0 \text{ and } (g - h)(1) = 0.$$

Suppose that there exists a t $(0, 1)$ for which $(g - h)(t\ ) > 0$. By the MVT

(2) Is T compact? Prove your answer.

Solution: One method is to use the spectral theorem. Since T is bounded and self-

Problem 4 Solution:

(a) Show that there is a bounded linear map $J : H \to H$ such that Jx is the unique element satisfying $A(x, y) = \langle Jx, y\rangle$ for all $y \in H$.

Solution: If x is fixed, then $y \to A(x, y)$ is linear (since A is bilinear), and it is bounded since we assume $|A(x, y)| \leq \beta \|x\| \cdot \|y\|$. Therefore, by the Riesz representation theorem, for a fixed x, we can write $A(x, y) = \langle z, y\rangle$ for all y, for some $z \in H$, and this z is unique. This z is some function of x, so let's write it as $z = J(x)$. Fixing y now, we know $x \to A(x, y)$ is linear in x (and bounded), and it is equivalent to writing $x \to \langle J(x), y\rangle$. Therefore $J(x)$ must be bounded and linear.

(b) Show that $\alpha \|x\| \leq \|Jx\|$.

Solution: If $x = 0$, this follows trivially, so assume $x \neq 0$ from now on. We assume $\alpha \|x\|^2 \leq |A($

Problem 5 Solution: It does converge. You could probably prove this via Fubini's theorem (with the counting measure, to turn the sum into an integral), but here's a method with the monotone convergence theorem ("MCT"). Define

$$f_n(x) = \begin{cases} 0 & x < n \\ f(x) & x \geq n. \end{cases}$$

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_n^{\infty} f(x)\,dx &= \lim_{N\to\infty} \sum_{n=1}^{N} \int_n^{\infty} f(x)\,dx \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \int_{\mathbb{R}} f_n(x)\,dx \\
&= \lim_{N\to\infty} \int_{\mathbb{R}} \sum_{n=1}^{N} f_n(x)\,dx \quad \text{(linearity of integral)} \\
&= \int_{\mathbb{R}} \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x)\,dx \quad \text{(this is monotone in } N \text{ due to non-negativity, so use } \boxed{\text{MCT}}\text{)} \\
&= \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)\,dx \\
&= \int_{\mathbb{R}} x\, f(x)\,dx \quad \text{(by observation)} \\
&\leq \int_{\mathbb{R}} x f(x)\,dx
\end{aligned}$$

so it is a bounded, monotone sequence, hence it converges.

You can also use Fubini's theorem, writing the sum as an integral via the counting measure (see example 12.30 in the book). Fubini applies, since both the counting measure and Lebesgue measure are σ-finite. Everything is non-negative, so we can ignore the absolute value. Thus Fubini says that if the integral works in either order, the integral exists, and it's the same value in either order. So you can jump straight to the last three lines above, and then since it is bounded, concluded the equality with the sum in the original order, and make the same conclusion.

You can also use DCT to interchange the limit, since $\sum_{n=1}^{N} f_n(x) \leq x\, f(x)$, and we know $xf(x$