

Applied Analysis Preliminary Exam

10.00am{1.00pm, August 21, 2017 (Draft v7, Aug 20)

Instructions. You have three hours to complete this exam. Work all the problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Let F be a family of equicontinuous functions from a metric space $(X; d_X)$ to a metric space $(Y; d_Y)$. Show that the completion of F is also equicontinuous.

(b) Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $C([0; 1])$. Let $\| \cdot \|$ be the sup norm. Suppose that, for all n , we have

$$\begin{aligned} \|f_n\| &\leq 1, \\ f_n &\text{ is differentiable, and} \\ \|f_n'\| &\leq M \text{ for some } M > 0. \end{aligned}$$

Show that the completion of $\{f_n\}_{n=1}^\infty$ is compact, and therefore that it has a convergent subsequence.

Problem 2:

Show that there is a continuous function u on $[0; 1]$ such that

$$u(x) = x^2 + \frac{1}{8} \int_0^x \sin(u^2(y)) dy;$$

Problem 3:

Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{f(x) |x|^n}{1+x^2} dx$$

exists and equals $\int_{\mathbb{R}} f(x) dx$.

Problem 4:

Let $K : L^2([0;1]) \rightarrow L^2([0;1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y) dy:$$

This operator can be shown to be compact by using the Arzela-Ascoli Theorem. For this problem, you may take compactness as fact.

- (a) Find the adjoint operator K^* of K .
- (b) Show that $\|K\|^2 = \|K^*K\|$.
- (c) Show that $\|K\| = 2^{-1/2}$. (Hint: Use part (b).)
- (d) Prove that

$$K^n f(x) = \frac{1}{(n-1)!} \int_0^x f(y) (x-y)^{n-1} dy:$$

- (e) Show that the spectral radius of K

Problem 1 Solution:

(a) This part is almost trivial. It is just here to help with part (b).

Recall that F being equicontinuous means that, for any $\epsilon > 0$, $\delta > 0$ such that $d_X(x; y) < \delta \implies d_Y(f(x); f(y)) < \epsilon$ holds $\forall f \in F$.

To show equicontinuity of the completion, we need only worry about the additional included functions. Let g be a function in the completion of F that was not in F to begin with. Since F is dense in the completion, we can find an $f \in F$ that is arbitrarily close to g . In particular, choose $f \in F$ such that $d_Y(f(x); g(x)) < \epsilon/3$ $\forall x \in X$.

Let $\epsilon > 0$. Note that

$$d_Y(g(x); g(y)) \leq d_Y(g(x); f(x)) + d_Y(f(x); f(y)) + d_Y(f(y); g(y)):$$

Since $f \in F$, we can find a $\delta > 0$ such that $d_Y(f(x); f(y)) < \epsilon/3$ and we are done. This gives us $d_X(x; y) < \delta \implies d_Y(g(x); g(y)) < \epsilon$.

(b) We will use the Arzela-Ascoli Theorem: Let K be a compact metric space. A subset of $C(K)$ is compact if and only if it is closed, bounded, and equicontinuous.

The completion of $\{f_n\}$ is, by definition, closed

By the assumptions of this problem, we also have that the completion of $\{f_n\}$ is bounded.

It remains to show that the completion of $\{f_n\}$ is equicontinuous.

Take $\epsilon > 0$. Fix n . By the Intermediate Value Theorem, we know that, $\forall x; y \in [0; 1]$, there exists a c between x and y such that $f_n(x) - f_n(y) = f'_n(c)(x - y)$.

Thus, we have that $|f_n(x) - f_n(y)| \leq M|x - y|$.

Define $\delta = \epsilon/M$. We then have

$$|x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon:$$

Note that this is independent of the choice of n .

Thus, the family of functions $\{f_n\}$ is equicontinuous.

By part (a) we know then that the completion of this family is equicontinuous.

By the Arzela-Ascoli Theorem, we then have that the completion of $\{f_n\}$ is compact, as desired.

Problem 2 Solution:

We will use the Contraction Mapping Theorem: If $T : X \rightarrow X$ is a contraction mapping on a complete metric space $(X; d)$, then T has exactly one fixed point. (i.e. There is exactly one $x \in X$ such that $T(x) = x$.)

Define

$$Tu(x) = x^2 + \frac{1}{8} \int_0^x \sin(u^2(y)) dy:$$

Note that T maps $C([0; 1])$ functions to $C([0; 1])$ functions. Since $C([0; 1])$ is complete with respect to the sup norm $\|f - g\|_\infty$, the contraction mapping theorem applies. u.504how the

By the mean value theorem, we know that there is some $s \in [0; 1]$ such that

$$\frac{\sin u}{u} - \frac{\sin v}{v} = \cos s (u - v)$$

so

$$|\sin u(y) - \sin v(y)| \leq |u(y) - v(y)|$$

So, we have that

$$\begin{aligned} \|Tu - Tv\|_{\infty} &= \sup_{x \in [0; 1]} \int_0^x |u^2(y) - v^2(y)| dy \\ &= \sup_{x \in [0; 1]} \int_0^x |u(y) + v(y)| |u(y) - v(y)| dy \\ &\leq \sup_{x \in [0; 1]} \int_0^x (|u(y)| + |v(y)|) |u(y) - v(y)| dy \end{aligned}$$

Since u and v are assumed to be continuous functions on the closed bounded interval $[0; 1]$, they are bounded on $[0; 1]$. Suppose that they are bounded by $M > 0$. Then

$$\begin{aligned} \|Tu - Tv\|_{\infty} &\leq \sup_{x \in [0; 1]} \int_0^x |u(y) + v(y)| |u(y) - v(y)| dy \\ &\leq \frac{M}{4} \int_0^1 |u(y) + v(y)| |u(y) - v(y)| dy \\ &= \frac{M}{4} \|u - v\|_{\infty} \end{aligned}$$

This may or may not be a contraction, depending on the value of M , but, we are trying to show **existence** of a solution in $C([0; 1])$. If we can show existence of a solution on some subset of $C([0; 1])$, we are done. So, let's limit our search to the set of continuous functions on $[0; 1]$ that are bounded, in the uniform norm, by some fixed constant M such that $M < 4$. Fix such an M and define the space

$$C := \{u \in C([0; 1]) : \|u\|_{\infty} \leq M\} \subset C([0; 1])$$

Note that this is a closed (and non-empty!) subset of the complete $C([0; 1])$ and is therefore complete. Furthermore, M can be chosen so that $T : C \rightarrow C$.

Thus, we have a contraction mapping on a complete space (that is a subspace of the space of interest). By the Contraction Mapping Theorem, there exists a unique fixed point $u \in C$ ($C([0; 1])$), which is a solution to the problem.

Problem 3 Solution:

This is trivial if $\int_{\mathbb{R}} |f(x)| dx = 0$. So, let us consider the case where $\int_{\mathbb{R}} |f(x)| dx > 0$.

Note that

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2} dx \leq \left(\int_{\mathbb{R}} |f(x)| dx \right)^n \int_{\mathbb{R}} \frac{1}{1+x^2} dx = \left(\int_{\mathbb{R}} |f(x)| dx \right)^n \cdot \pi \quad (S1)$$

as $n \geq 1$.

On the other hand, by definition of $\int_{\mathbb{R}} |f(x)| dx$, for any $0 < \epsilon < \int_{\mathbb{R}} |f(x)| dx$, there exists an $A \in \mathbb{R}$ (with positive Lebesgue measure) such that $\int_A |f(x)| dx > \epsilon$.

Thus, we have

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2} dx \geq \int_A \frac{|f(x)|^n}{1+x^2} dx \geq \left(\int_A |f(x)| dx \right)^n \int_A \frac{1}{1+x^2} dx$$

Note that $\int_A \frac{1}{1+x^2} dx$ is strictly positive. Call it $c > 0$.

For all n , we now have

$$\int_{\mathbb{R}} \frac{f(x)j^n}{1+x^2}$$

(c)

so we have that

$$[(n-1)!]^{1/n} \left(\frac{\rho}{2}\right)^{1/n} (n-1)^{1-1/(2n)} e^{1/n-1}$$

which goes to 1 as $n \rightarrow \infty$.

In conclusion, the spectral radius is

$$r(K) = \lim_{n \rightarrow \infty} \|K^n\|^{1/n} = 0;$$

as desired.

Problem 5 Solution: