Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2017 (Draft v7, Aug 20)

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Let $\mathcal{F}$ be a family of equicontinuous functions from a metric space $(X; d_X)$ to a metric space $(Y; d_Y)$. Show that the completion of $\mathcal{F}$ is also equicontinuous.

(b) Let $(f_n)_{n\geq 1}$ be a sequence of functions in $C([0;1])$. Let $\|\cdot\|_\infty$ be the sup norm. Suppose that, for all n, we have

- $\|f_n\|_\infty \leq 1$,
- f_n is differentiable, and
- $\|f'_n\|_\infty \leq M$ for some $M \geq 0$.

Show that the completion of $\{f_n\}_{n\geq 1}$ is compact, and therefore that it has a convergent subsequence.

Problem 2:

Show that there is a continuous function u on $[0;1]$ such that

$$u(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\,dy.$$

Problem 3:

Let $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n}$$

exists and equals $\|f\|_\infty$.

Problem 4:

Let $K : L^2([0,1]) \to L^2([0,1])$ be the integral operator defined by

$$Kf(x) = \int_0^x f(y)\, dy.$$

This operator can be shown to be compact by using the Arzela-Ascoli Theorem. For this problem, you may take compactness as fact.

(a) Find the adjoint operator K^* of K.

(b) Show that $\|K\|^2 = \|K^*K\|$.

(c) Show that $\|K\| = 2/\pi$. (Hint: Use part (b).)

(d) Prove that

$$K^n f(x) = \frac{1}{(n-1)!} \int_0^x f(y)(x-y)^{n-1}\, dy.$$

(e) Show that the spectral radius of K