Applied Analysis Preliminary Exam (Hints/solutions)

10.00am–1.00pm, August 18, 2016

Problem 1:

(a) Prove (using the comparison test or the Weierstrass-M test) the "Cauchy root test": If $C = \limsup_{n \to \infty} |a_n|^{1/n} < \infty$, then the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges uniformly if $|z| < 1/C$ and diverges if $|z| > 1/C$.

(b) What does this result say about the series

$$\sum_{n=0}^{\infty} 2^n \sin(n) z^n$$

Solution/Hint: Mainly straightforward.

Problem 2: (The two sub-problems are unrelated)

(a) One of the requirements of the Weierstrass Approximation Theorem is that the function to be approximated is continuous on a closed and bounded interval I. Show that the Approximation Theorem does not hold if we replace I by a bounded open interval $(a; b)$ by showing that if $f(x) = 1/(b - x)$, then $f : (a; b) \to \mathbb{R}$ cannot be uniformly approximated by polynomials.

Solution/Hint: Mainly straightforward. Do not use Taylor series; even if the Taylor series doesn't approximate the function, that doesn't prove that there is no other polynomial which approximates the function.

(b) Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis for a Hilbert space H and $A : H \to H$ a bounded linear operator. If

$$\lim_{n \to \infty} \sup_{\substack{u \perp \{e_1, \ldots, e_n\} \\ u \neq 0}} \frac{\|Au\|}{\|u\|} = 0$$

then prove A is a compact operator.

Solution/Hint: The general idea is to approximate A with a finite-rank operator, which is therefore compact, and then show that these approximations converge uniformly to