Applied Analysis Preliminary Exam

10.00am–1.00pm, August 18, 2016

Instructions: You have three hours to complete this exam. Work all five problems; each is worth 20 points. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are being asked to prove such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name.

Problem 1:

(a) Prove (using the comparison test or the Weierstrass-M test) the "Cauchy root test": If $C = \limsup_{n\to\infty} |a_n|^{1/n} < \infty$, then the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges uniformly if $|z| < 1/C$ and diverges if $|z| > 1/C$.

(b) What does this result say about the series

$$\sum_{n=0}^{\infty} 2^n \sin(n) z^n$$

Problem 2: (The two sub-problems are unrelated)

(a) One of the requirements of the Weierstrass Approximation Theorem is that the function to be approximated is continuous on a closed and bounded interval I. Show that the Approximation Theorem does not hold if we replace I by a bounded open interval (a, b) by showing that if $f(x) = 1/(b - x)$, then $f : (a, b) \to \mathbb{R}$ cannot be uniformly approximated by polynomials.

(b) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space H and $A : H \to H$ a bounded linear operator. If

$$\lim_{n\to\infty} \sup_{\substack{u \perp \{e_1,\dots,e_n\} \\ u\neq 0}} \frac{\|Au\|}{\|u\|} = 0$$

then prove A is a compact operator.

Problem 3: Let (X, d) be a complete metric space. A function $f : X \to X$ is said to be a contraction if there exists $c < 1$ such that $d(f(x), f(y)) \le c \cdot d(x, y)$ for all $x, y \in X$. A function f is said to be a *weak contraction* if

$$d(f(x), f(y)) < d(x, y) \quad \forall\, x \neq y;\ x, y \in X.$$

Note that a weak contraction is Lipschitz continuous with Lipschitz constant 1.

(a) Prove the following variant of the contraction mapping theorem: if f is a *weak contraction* and the space X is *compact*, then f has a unique fixed point in X. Hint: consider the function $g(x) = d(x, f(x))$ over X.

(b) Let X be $\ell^2(\mathbb{N})$ and $f(x) = L(x) + b$ be an affine function defined by mapping $L : x \mapsto y$ where $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ with $y_n = (1 - \frac{1}{n})x_n$, and $b = (\frac{1}{n})_{n\in\mathbb{N}}$. Prove or disprove that f has a fixed point in X. Does your answer change if X is the closed unit ball in $\ell^2(\mathbb{N})$?

Problem 4: Let H be an infinite dimensional Hilbert space, and M a subspace of H. If $\varphi \in M^*$, prove φ has a *unique* norm-preserving extension to a bounded linear functional on all of H, and that this extension vanishes on $M^\perp$.

Problem 5: State and prove Fatou's Lemmahis 9Fatou's a6plete 9F