

A *A*
10.00AM–1.00PM, AUGUST 21, 2012

INSTRUCTIONS.

Solution sketch:

1. Integrating the differential equation, we get

$$v(x) = \frac{1}{4} + \frac{1}{4} \int_0^x \sin(s + v^2(s)) ds; \quad (1)$$

Let $\| \cdot \|_u = \sup_{t \in [0,1]} |u(t)|$ denote the uniform norm, and define the set

$$X = \{ u \in C[0;1] : u(0) = \frac{1}{4} \text{ and } \|u\|_u \leq 1 \};$$

The set X combined with the uniform norm is a metric space. Now define the operator

$$[T](x) = \frac{1}{4} + \frac{1}{4} \int_0^x \sin(s + (u(s))^2) ds;$$

the IVP can then be written as a fixed point problem $Tv = v$.

First observe that if $u \in X$, then T

• Let I denote the line in the complex plane $I = \{z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \in [-\frac{\pi}{2}; \frac{\pi}{2}]\}$.

- (a) Set $\lambda = \alpha + i$ where α and β are real. Set $C = \sup_{x \in I} |\lambda + z| = \sqrt{(\frac{\pi}{2} + |\alpha|)^2 + 1}$. Since $|(Au)(x)| \leq C|u(x)|$ for all x , we get $\|A\| \leq C$. For the converse, suppose that $\alpha \geq 0$ (the proof for $\alpha < 0$ is analogous). Set $u = \chi_{[-n, +1]}$. Then $\|u\| = 1$ and

$$\|Au\|^2 = \int_{-n}^{+1} |(\alpha + \arctan(x))|^2 dx = \int_{-n}^{+1} (\alpha + \arctan(x))^2 dx \geq (\alpha + \arctan(n))^2 \rightarrow C:$$

- (b) We have

$$(Au; v) = \int_{\mathbb{R}} \overline{u(x)} v(x) dx + \int_{\mathbb{R}} \arctan(x) \overline{u(x)} v(x) dx \quad (2)$$

$$(u; Av) = \int_{\mathbb{R}} \overline{u(x)} v(x) dx + \int_{\mathbb{R}} \arctan(x) \overline{u(x)} v(x) dx: \quad (3)$$

We see that A is self-adjoint if and only if α is real.

- (c) Suppose that $Au = 0$. Then $(\alpha + \arctan(x)) u(x) = 0$ almost everywhere. This can happen only if $u = 0$. It follows that A is one-to-one for all α .
- (d) If $\alpha \in I$, then set $\delta = \min_{x \in I} |\alpha - z| = \text{dist}(I; \alpha)$. Since I is closed, $\delta > 0$. Clearly $\|Au\| \geq \|u\|$, so A has closed range. To prove the converse, we will use that since A is one-to-one for all α , it has closed range if and only if it has a continuous inverse. Suppose first that $\alpha \in (-\frac{\pi}{2}; \frac{\pi}{2})$. Set $I_n = (\tan(\alpha) - 1/n; \tan(\alpha) + 1/n)$ and $u = \chi_{I_n}$. Then $\lim_{n \rightarrow 0} \|Au\|/\|u\| = 0$, so A does not have a bounded inverse. If $\alpha = \pm \frac{\pi}{2}$, then use $u = \chi_{[\alpha, +1]}$ to show that A is

