Remember to write your name! You are allowed to use a calculator. You are not allowed to use the textbook, your notes, the internet, or your neighbor. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

Name:

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1. (30 points) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." If the statement seems neither true nor false but rather incoherent, raise your hand. No justification is necessary. **Students in 4720 can pick 5 out of 6 questions to answer. Students in 5720 must answer all.**

_____(a) If a matrix A is normal then the eigenvalues of the perturbed matrix $A + E$ are all within a distance $\|E\|_3$ of the eigenvalues of A.

False This is the Bauer-Fike theorem. You can use any p-norm. But in the 3-norm the condition of the eigenvector basis is not necessarily 1, as it would be in the 2-norm.

_____(b) Let be a simple eigenvalue of A with left and right eigenvectors y and x, each of which are 2-norm unit vectors, and let $A + E$ be the perturbed matrix where $\|E\|_2 =$. True or False: The perturbed matrix will have an eigenvalue within a distance of approximately $/|y\ x|$ from , for small-enough .

True

_____(c) Let A be a diagonalizable matrix with approximate eigenvalue/eigenvector pair $(; x)$. True or false: $(; x)$ is an exact eigenvalue/eigenvector pair for a perturbed matrix $A + E$ where $\|E\|_2 \quad \|Ax \quad x\|_2$.

False This would be true for *normal* matrices, but the correct statement for non-normal matrices includes the condition number of the eigenvector basis.

_____(d) Suppose that A is $n \quad n$ with LU factorization $PA = LU$. True or false: The matrix $UP^T L$ has the same eigenvalues as A.

True The matrices are similar:

$$UP^T L = L^{-1} PAP^T L.$$

L is always invertible (even if A is not) because it is lower triangular with ones on the diagonal.

_____(e) Let and x be an eigenvalue/eigenvector pair for A. True or false: The matrix $A \quad xx \ /\|x\|_2^2$ has eigenvector x with eigenvalue 0.

True

_____(f) Let S be a nontrivial subspace that is invariant under a square matrix A. True or False: There is an eigenvector of A in S

2. (20 points) Suppose that you are given one eigenvalue/eigenvector pair of an $n \times n$ matrix A. Explain how you can reduce the problem of finding the remaining eigenvalues of A to finding the eigenvalues of an $n-1 \times n-1$ matrix. Show explicitly how to construct the $n-1 \times n-1$ matrix. Hint: Start by constructing an invertible matrix X whose first column is the eigenvector.

   Let X be an invertible matrix whose first column is the eigenvector. Then

$$X^{-1}AX = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & B \end{array}\right].$$

   A is similar to the RHS, which is block-upper triangular. The eigenvalues of A are therefore λ together with the eigenvalues of B, which is $n-1 \times n-1$. Kudos if you used a unitary similarity transform rather than just an invertible X.

3. Computing the SVD of a real $m \times n$ matrix A requires computing the eigenvalues and eigenvectors of A^TA and AA^T.

   (a) Let P and Q be real orthogonal matrices of size $m \times m$ and $n \times n$ respectively, and let $B = PAQ$. Show that the singular values of B are the same as the singular values of A.

   The singular values of A are the square roots of the eigenvalues of A^TA, and similarly for the singular values of B. Note that

$$B^TB = Q^TA^TAQ$$

   so B^TB is (orthogonally-)similar to A^TA, and they therefore have the same eigenvalues.

   (b) Let v be an eigenvector of B^TB. How is it related to the corresponding eigenvector of A^TA?

   The above analysis shows that if v is an eigenvector of B^TB, then Qv is an eigenvector of A^TA.

   (c) It is possible to choose P and Q such that B is bi-diagonal (nonzeros immediately above the diagonal). Prove that B^TB and BB^T are tridiagonal (you may cite any relevant theorem from class).

   The banded-matrix-multiplication theorem shows that multiplying a lower-bidiagonal and an upper-bidiagonal matrix yields a tridiagonal matrix.

4. (20 points)

5720 Only Let A be an $n \quad n$ diagonalizable matrix with eigenvalues satis[redacted] $::: = \;_k$ with $j \;_k j > j \;_{k+1} j \quad ::: \quad j \;_n j$. Show that the vectors generated by [redacted] method will converge to an eigenvector of A (under standard assumptions o[redacted] ing vector).

Let the eigenvecto[redacted]
$x_0 = c_1 v_1 + \quad +$ [redacted]

$A^p x$ [redacted]

A^p [redacted]

$c_k +$ $+ \;\; +$ [redacted]