Synchronization in large directed networks of coupled phase oscillators

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We study the emergence of collective synchronization in large directed networks of heterogeneous oscillators by generalizing the classical Kuramoto model of globally coupled phase oscillators to more realistic networks. We extend recent theoretical approximations describing the transition to synchronization in large undirected networks of coupled phase oscillators to the case of directed networks. We also consider the case of networks with mixed positive-negative coupling strengths. We compare our theory with numerical simulations and find good agreement. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2148388]

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I. INTRODUCTION

The classical Kuramoto model[13,14] describes a collection of globally coupled phase oscillators that exhibit a transition from incoherence to synchronization as the coupling strength is increased past a critical value. Since real world networks typically have a more complex structure than all-to-all coupling,[15,16] it is natural to ask what effect interaction structure has on the synchronization transition. In Ref. 12, we studied the Kuramoto model allowing general connectivity of the nodes, and found that for a large class of networks there is still a transition to global synchrony as the coupling strength exceeds a critical value k_c. We found that the critical coupling strength depends on the largest eigenvalue of the adjacency matrix A describing the network connectivity. We also developed several approximations describing the behavior of an order parameter measuring the coherence past the transition. This past work was restricted to the case in which $A_{nm}=A_{mn}\geq 0$, that is, undirected networks in which the coupling tends to reduce the phase difference of the oscillators.

Most networks considered in applications are directed,[15,16] which implies an asymmetric adjacency matrix, $A_{nm}\neq A_{mn}$. Also, in some cases the coupling between two oscillators might drive them to be out of phase, which can be represented by allowing the coupling term between the two oscillators to be negative, $A_{nm}<0$. The effect that the presence of directed and mixed positive-negative connections can have on synchronization is, therefore, of interest. Here we show how our previous theory can be generalized to account for these two factors. We study examples in which either the asymmetry of the adjacency matrix or the effect of the negative connections are particularly severe and compare our theoretical approximations with numerical solutions.

This paper is organized as follows. In Sec. II we review the results of Ref. 12 for undirected networks with positive man he erogeneo co pled pha e o cilla or . Thi i a ion can be modeled b he eq a ion

$$\dot{}_n = {}_n + k \sum_{m=1}^{N}$$

A eraging o er he freq encie , one ob ain he *frequency distribution approximation* (FDA)

$$r_n = k \sum_m A_{nm} r_m \int_{-1}^{1} g(zkr_m) \sqrt{1 - z^2} dz. \quad (13)$$

The al e of he cri ical co pling reng h can be ob- ained from he freq enc di rib ion appro ima ion b le - ing $r_n \to 0^+$, prod cing

$$r_n^{(0)} = \frac{k}{k_0} \sum_m A_{nm} r_m^{(0)}, \quad (14)$$

here k_0 2/ $g(0)$. The cri ical co pling reng h h corre pond o

$$k_c = \frac{k_0}{\ }, \quad (15)$$

here i he large eigen al e of he adjacenc ma ri A and $r^{(0)}$ i propor ional o he corre ponding eigen ec or of A. B con idering per rba ion from he cri ical al e a $r_n = r_n^{(0)} + r_n$, e panding $g(zkr_m)$ in Eq. (13) o econd order for mall arg men , m l ipl ing Eq. (13) b $r_n^{(0)}$ and m-ming o er n, e ob ained an e pre ion for he order param- e er pa he ran i ion alid for ne ork i h rela i el ho-mogeneo degree di rib ion [17]

$$r^2 = \left(\frac{1}{k_0^2} \right) \left(\frac{k}{k_c} - 1 \right) \left(\frac{k}{k_c} \right)^{-3}, \quad (16)$$

for 0 (k/k_c) 1 ≪ 1, he2r4115.078hj/F6.768598406.2813Tm2764444111..9.9789j/F599Tc-307.9h37859.di rib ion 6.913Tm2764444

$$r = \square_{n=1}^{N} r$$

he adjacenc ma ri i independen l cho en o be 1 i h probabili s and 0 i h probabili 1 s, and he diagonal elemen are e o ero. E en ho gh he ne ork con- r c ed in hi a i direc ed, for mo node $d_n^{\rm in} \approx d_n^{\rm o}$. For $N=1500$ and $s=2/15$, Fig. 1 a ho he a erage of he order parame er r^2 ob ained from n merical ol ion of Eq. 1 a eraged o er en reali a ion of he ne ork and fre- q encie riangle , he freq enc di rib ion appro ima- ion FDA, olid line , and he mean eld heor MFT, long da hed line a a f nc ion of k/k_c, here he re l for he FDA and he MFT are a eraged o er he en ne ork real- i a ion no e, ho e er, ha he FDA and he MFT do no depend on he freq enc reali a ion . The per rba ion heor Eq. 16 agreed i h he freq enc di rib ion ap- pro ima ion and a lef o for clari . The error bar cor- re pond o one andard de ia ion of he ample of en real- i a ion . We no e ha he larger error bar occ r af er he ran i ion. When he al e of he order parame er are a er- aged o er en reali a ion of he ne ork and he freq encie , he re l ho er good agreemen i h he freq enc di rib ion appro ima ion and he direc ed mean eld heor .

In order o d ho ell o r heor de cribe ingle reali a ion , e ho in Fig. 1 b he order parame er r^2 ob ained from n merical ol ion of Eq. 1 for a par ic lar reali a ion of he ne ork and freq encie bo e , he ime a eraged heor hor da hed line , and he freq enc di - rib ion appro ima ion olid line a a f nc ion of k/k_c. A can be ob er ed from he g re, in con ra i h he ime a eraged heor , he freq enc di rib ion appro ima ion de ia e from he n merical ol ion bo e b a mall b no iceable amo n . Thi beha ior i ob er ed for he o her reali a ion a ell. We no e ha he FDA and MFT re l are ir all iden ical for all en reali a ion . On he o her hand, he TAT and he re l from direc n merical ol ion of Eq. 1 ho dependence on he reali a ion. Since he FDA and MFT incorpora e he reali a ion of he connec- ion A_{nm}, b no he freq encie , e in erpre he ob er ed reali a ion dependence of he TAT and he direc ol ion of Eq. 1 a indica ing ha he la er dependence i d e prima- ril o c a ion in he reali a ion of he freq encie ra her han o c a ion in he reali a ion of A_{nm}.

No e ha for o r e ample $N=1500$ and $s=2/15$ implie ha on a erage e ha e $d^{\rm in} \approx d^{\rm o} \approx 200$. Th for compari- on p rpo e , e genera ed an ndirec ed ne ork a fol- lo : S ar ing i h=a Eq.a9F54825ek-2 i h TDfrem reali a 4direc re , a in he ndirec ed ca e, he al e of he a erage of he order parame er ob ained from n merical ol ion of Eq. 1 . The direc ed per rba ion heor gi e a good appro i- ma ion for mall al e of k clo e o k_c, a e pec ed. On he o her hand, he direc ed mean eld heor predic a ran i- ion poin hich i maller han he one ac all ob er ed.

When n merically ol ing Eq. 32 b i era ion of Eq. 33, on ome occa ion a period o orbi a fo nd in ead of he de ired ed poin . If e deno e he lef hand ide of Eq. 33 b z_n^{j+1} and he righ hand ide b $f(z_n^j)$, e fo nd ha con ergence o a ed poin a facili a ed b replacing he righ hand ide b $(z_n^j + f(z_n^j))/2$ and nding he ed poin of hi modi ed em.

In hi e ample, a lo co pling reng h ro ghl $k/k_c \lesssim 4$, here k_c i comp ed from Eq. 37 he order parame er comp ed from n merical ol ion of Eq. 1 i maller han ha ob ained from he TAT and FDA. A k increa e , ho e er, he TAT and FDA heorie cap re he a mp o ic al e of he order parame er r. We no e ha in hi ca e he a mp o ic al e i larger han ha corre ponding o pha e locking i.e., he one ob ained b e ing $_n = 0$ in Eq. 35, r 0.54 0.46 = 0.08, hich e indica e b a hori on al do -da hed line in Fig. 4, and m ch maller han $r = 1$, he al e corre ponding o no fr ra ion i.e., $_n$ $_m$ = 0 for A_{nm} 0 and for A_{nm} 0 in Eq. 35. The mall cale of he hori on al a i i d e o he fac ha e are plo ing r^2, and o o r de ni ion of he order parame er hich a ign a al e of 1 o a nonfr ra ed con g ra ion. The mall al e of he order parame er indica e a rong fr ra ion.

We no e ha in hi e ample, in con ra i h he e ample di c ed o far, here i aria ion in he al e of he order parame er predic ed b he FDA for differen reali a ion of he ne ork. Thi indica e ha , a he e pec ed al e of he co pling reng h A_{nm} become mall i.e., q 1/2 mall, c a ion d e o he reali a ion of he ne ork become no iceable. Al ho gh he al e predic ed b he FDA and TAT depend on he reali a ion of he ne ork and freq encie , e no e for k/k_c 6 ha he e al e rack he al e ob er ed for he n merical im la ion of he corre ponding reali a ion. A an ill ra ion of hi , e plo in Fig. 5 he al e of r^2 ob ained from he TAT ar and he al e of r^2 ob ained from he FDA diamond er he al e ob ained from n merical ol ion of Eq. 1 for k/k_c = 8. Each poin corre pond o a gi en reali a ion of he ne ork, i h re l a eraged o er en reali a ion of he freq encie . The ellip e rro nding he ar TAT da a ha e er ical and hori on al half- id h corre ponding o he andard de ia ion of r^2 TAT and r^2 im la ion for he en freq enc reali a ion . The half- id h of he hori on al bar on he diamond FDA da a indica e he andard de ia ion of r^2 im la ion la ion in ne ork i h a m, ch larger n, mber of connec- ion per node, a he effec of , c , a ion o, ld likel be red, ced.

of he non ero en rie being cho en randoml □e.g., in he mme ric ca e, he po i ion of he non ero en rie i cho en hen con r c ing he ne ork ing he con g ra ion model□, and heir al e being al o de ermined randoml from a gi en probabili di rib ion □e.g., 1 i h probabili q and 1 i h probabili 1 q□. O r in ere i foc ed on he gap be een he large real eigen al e □if here i one□ and he large real par of he o her eigen al e . In Ref. 23 he pec r m of cer ain large par e ma rice i h a erage eigen al e 0 and ro m $\Box_{m=1}A_{nm}=1$ a de cribed and a he ri ic anal ical approach a propo ed. U ing re l for ma rice i h ero mean Ga ian random en rie ,[24] Ref. 23 predic ha he pec r m of he non-Ga ian random ma- rice he con ider con i of a ri ial eigen al e =1 i h he remaining eigen al e di rib ed niforml in a circle cen ered a he origin of he comple plane i h radi

$$= \Box\overline{N} \;, \qquad □A1□$$

here ^{2} i he ariance of he en rie of he ma ri . We nd ha hi approach al o cceed in de cribing he pec r m of he ma rice in o r e ample . In o r ca e, he diagonal en rie are 0, o ha he a erage eigen al e i al o 0 a in Ref. 23. We nd ha here i al a a large real eigen al e appro ima el gi en b he mean eld al e

$$= \Box\tilde{d}^2\Box/\Box\tilde{d}\Box \qquad □A2□$$

□ ee Ref . 12 and 25□, here $\tilde{d}_n=\Box_{m=1}^{N}A_{nm}$ and $\Box\tilde{d}^2\Box=\Box_{n=1}^{N}\tilde{d}_n^2$, hich in he ca e con idered in Ref. 23 red ce o =1. We al o n mericall con rm ha he remaining eigen al e are niforml di rib ed in a circle of radi a de cribed in Ref. 23. Thi i ill ra ed in Fig. 6.

Th for N if here i a gap of i e be- een he large real eigen al e and real par of he re of he eigen al e pec r m. U ing Eq . □A1□ and □A2□ i can be ho n ha , for ne ork i h large eno gh n mber of con- nec ion per node or i h eno gh po i i e □or nega i e□ bia in he co pling reng h, here i a ide epara ion be een he large eigen al e and he large real par of he remain- ing eigen ec or . For mme ric ma rice , imilar re l ap- pl □i.e., he b lk of he pec r m of he ma ri A can be appro ima el ob ained a de cribed abo e ing Wigner' emicircle la □.

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