Average exit time for volume-preserving maps

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For a volume-preserving map, we show that the exit time averaged over the entry set of a region is given by the ratio of the measure of the accessible subset of the region to that of the entry set. This result is primarily of interest to show two things: First, it gives a simple bound on the algebraic decay exponent of the survival probability. Second, it gives a tool for computing the measure of the accessible set. We use this to compute the measure of the bounded orbits for the Hénon quadratic map. @S1054-1500~97!00101-8#

One important aspect of chaos in conservative dynamical systems is that chaotic and regular regions are intermixed in an intricate way in the phase space. This can have important implications for transport properties of these systems. For example, orbits of an area-preserving map are eternally trapped in a region if they are enclosed by an invariant circle, but can leak through destroyed circles cantori . In this paper we obtain an effective technique for computing the fraction of trapped orbits. The result, applied to the Hénon map, shows that the trapped fraction depends in an intricate way on the structure of the chaotic set.

I. INTRODUCTION

In this paper we study the time of escape for orbits that begin in a specified region A under the dynamics of a map f on a phase space M. We discuss the *exit time*, the time for a point to first exit the set, and the *transit time*, the time for a point to traverse the set. The *exit time distribution* is the probability distribution of exit times. Our primary goal is to use this distribution to probe the trapped invariant set. The trapped set is generally quite difficult to compute, and is of interest because its boundaries are extremely ''sticky'' [1–13] and so even untrapped orbits feel its influence. Our results apply to volume-preserving maps in any dimension where the mechanisms for trapping and escape are much less understood.[14]

The theoretical result obtained in this paper is not new—it was essentially obtained by Kac in 1947,[15,16] and he even quotes earlier results of Birkhoff ~1931! and Smoluchowski ~1916!. Kac was studying the mean first return time to a region in a bounded phase space for an ergodic system; this can be called the mean Poincaré recurrence time or Poincaré cycle. He obtains his result as a consequence of the Poincaré recurrence theorem.[16] We reformulate the result for nonergodic systems, and show how it can also be obtained by considering the mean first exit time.

In his 1957 lecture Kac[16] remarks that the mean recurrence time is ''the only quantity which is tractable for general dynamical systems;'' however, he abandons it as ''relatively useless.'' We will not take this admonition to heart, and use our formulation to compute the measure of the trapped orbits.

As an example, consider Hénon's area-preserving, quadratic map, which we write as

H:~x,y!→~

there is a critical golden circle, obtaining a5 0.34 up to 10^5 iterates. Though algebraic decay has been observed in many Hamiltonian systems and symplectic maps ~indeed, whenever there are elliptic orbits the decay appears to be algebraic!, the exponent for the decay is apparently not universal.[21] One reason this might be so, according to Murray,[22] is that the self-similar limit is not reached for ''short'' time computations. The Poincaré recurrence distribution has also been computed for flows, for example by Zaslavsky and Tippet.[23] @Interestingly, they speculate that a relation like our Eq. ~13! holds.#

Rom-Kedar and Wiggins have emphasized the fact that

III. AVERAGE EXIT TIME

$\langle t$

In this paper we are interested in these transport times averaged over sets of initial conditions. For a function $g(z)$ we denote

$$\langle g \rangle_S \equiv \frac{1}{\mu(S)} \int_S g(z)\, d\mu.$$

Remarkably, there are some simple formulas for average transport times. The following lemma was stated for volume preserving flows (without its elementary proof) in Ref. 24 and is implicitly obtained for two-dimensional maps in Ref. 25.

Average Exit Time Lemma: The average exit (transit) time for incoming orbits is

$$\langle t^{+} \rangle_I = \langle t_{\mathrm{transit}} \rangle_I = \frac{\mu(A_{\mathrm{acc}})}{\mu(I)}. \tag{11}$$

Proof: Since $T_j \subset I$ are disjoint and cover almost all of I and the exit (and transit) time of the set T_j is j, the average exit time is given by

An almost identical expression was obtained by Kac (see Ref. 16, p. 66) though he does not make the interpretation about the accessible region. Rom-Kedar and Wiggins have also obtained this result.[25] Comparing Eq. (12) with the expression for $\langle t^{+}\rangle_I$ gives the lemma. Finally since $\mu(A_{acc}) \le \mu(A)$, it is clear that the sum in Eq. (12) converges. $\square$

Since $\mu(A_{acc})$ is finite, a simple consequence of this result is:

Corollary 1: The measure of the region with transit time t must decay faster than t^{-2}.

Furthermore, the lemma implies well-known results for the average return time obtained by Kac and Smoluchowski (Refs. 15–16). In our context these can be generalized to:

Corollary 2 (Smoluchowski): Suppose $\mu(M)=1$. The average first return time for points that escape in one step from $A \subset M$ is

$$\langle t_{return}\rangle_E = 1 + \frac{\mu(M_{acc}\backslash A)}{\mu(E)} = 1 + \frac{\mu(M_{acc}) - \mu(A)}{\mu(E)},$$

where M_{acc} is the subset of M that is accessible to orbits beginning in A.

Proof: Consider the set $M\backslash A$. Points enter it by escaping from A, so the entry set of $M\backslash A$ is $f(E)$. The corollary follows from the Lemma if we replace A_{acc} by $M_{acc}\backslash A$, and I by $f(E)$. We then add one to the result, since the return time to A is one larger than the transit time through M. $\square$

Corollary 3 (Kac): Suppose $\mu(M)=1$. The average first return time to a region $A \subset M$ is

$$\langle t_{return}\rangle_A = \frac{\mu(M_{acc})}{\mu(A)}, \tag{13}$$

where M_{acc} is the subset of M that is accessible to orbits beginning in A.

Proof: For points that stay in A for at least one step, the first return time is one. The remaining portion of A is its exit set E. We use Corollary 2 for the return time for these points. So the average first return time to A is

$$\langle t_{return}\rangle_A = \frac{1}{\mu(A)}\left[(\mu(A) - \mu(E)) \times 1 + \mu(E) \times \langle t_{return}\rangle_E\right].$$

This reduces to the promised result. $\square$

We can also easily compute the transit time averaged over A:

Corollary 4: The average transit time for points that do escape from A is

$\langle t_{transit}\rangle_{A_{acc}} = \frac{1}{\mu(A}$s5¬762.2hat96¬676¬Tf 23J pointoscoEq. 0.9SmoÖ777¬2¬526.8381.53923.5(

where a . 0 by Corollary 2, then we have

$$\mathrm{Prob}\tilde{}\,t^{1}\tilde{}\,I!\ 5\ k!\ ;\ k^{2\tilde{}a12!},$$

$$\mathrm{Prob}\tilde{}\,t^{1}\tilde{}\,I!>k!\ ;\ \mathrm{Prob}\tilde{}\,t^{1}\tilde{}\,A_{\mathrm{acc}}!\ 5\ k!$$

$$;\ \mathrm{Prob}\tilde{}\,t_{\mathrm{transit}}\tilde{}\,A_{\mathrm{acc}}!\ 5\ k!\ ;\ k^{2\tilde{}a11!},$$

$$\mathrm{Prob}\tilde{}\,t^{1}\tilde{}\,A_{\mathrm{acc}}!>k!\ ;\ k^{2a}.$$

V. EXAMPLES

Consider the linear, area-preserving, hyperbolic map

$$\begin{pmatrix} x8 \\ y8 \end{pmatrix} 5 \begin{pmatrix} l & 0 \\ 0 & l^{21} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where l . 1. Let A be the unit square $\$(x,y):0<x,y<1\%$. Then the entrance set is the rectangle $I\,5\,A\backslash f(A)$ $5\,\$(x,y):0<x<1, l^{21}, y<1\%$. It is easy to see that the transit time decomposition ~see Fig. 3! of I is

$$T_j\,5\,\$\tilde{}\,x,y!:l^{2j},\ x<l^{12j},\ \ l^{21},\ y<1\%.$$

So the measures of each of these regions are

$$m\tilde{}\,T_j!\ 5\ \frac{\tilde{}\,l\,2\,1!\tilde{}\,1\,2\,l^{21}!}{l^{j}}. \qquad \text{~18!}$$

These decay exponentially, as one would expect. The calculations needed for Eqs. ~9!, ~12!, and ~14! are derivatives of simple geometric sums, yielding

$$\sum_{j51}^{`} m\tilde{}\,T_j!\ 5\ \tilde{}\,1\,2\,l^{21}!\ 5\ m\tilde{}\,I!,$$

$$\sum_{j51}^{`} j\,m\tilde{}\,T_j!\ 5\ 1\ 5\ m\tilde{}\,A!,$$

$$\sum_{j51}^{`} j^{2}\,m\tilde{}\,T_j!\ 5\ \frac{l\,1\,1}{l\,2\,1},$$

as required by the lemma. Thus the average transport times are

$$\hat{}\,t^{1}\&_{I}\ 5\ \frac{l}{l\,2\,1},\quad \hat{}\,t_{\mathrm{transit}}\&_{A}\ 5\ \frac{l\,1\,1}{l\,2\,1},\quad \hat{}\,t^{1}\&_{A}\ 5\ \frac{l}{l\,2\,1}. \qquad \text{~19!}$$

These results are unchanged if we scale the size of A ~since the map is linear!

$$\mu(T_1) = \frac{1}{4}, \quad \mu(T_j) = \frac{1}{j(j^2 - 1)}\ j > 1.$$

The sums to get the average transport times are elementary telescoping sums, and again these verify Eqs. ~9! and ~12!. However, for this example the average transit time for A, Eq. ~14!, does not exist.

VI. BOUNDED ORBITS FOR THE HÉNON MAP

As a final example we use the average transit time, which is straightforward to compute, as an effective method to obtain the accessible area, which is not otherwise computable. Here we do this for the resonance zone of the Hénon map. The calculation involves several steps. First we find the points on the minimizing and minimax homoclinic orbits, z_m and z_h, that bound the lobe ~see Fig. 1!. We then construct the boundary of the entry set, by discretizing W^u and W^s, as graphs $y^u(x)$ and $y^s(x)$, to a resolution h 5 (xæ.75vera0.10Tj 6.

is exponentially small[28] and most of the resonance is filled with invariant curves.

Combining these results, using Eq. (11) gives the accessible area. In Fig. 7 we show the accessible fraction, $\mu(A_{acc})/\mu(A)$. The area that is inaccessible, which is identical to the measure of the bounded orbits is given by

$$\mu(A_i) = \mu(A) - \mu(A_{acc}) = \mu(A) - \mu(I)\langle t^{-1}\rangle_I .$$

This area is shown in Fig. 8; this figure is nearly identical to Fig. 4 of Ref. 1, but our method allows us to compute the results to much higher precision. The accuracy can be seen better in the next figure, which is the best representation of this information, Fig. 9. This shows the inaccessible fraction, $\mu(A_i)/\mu(A)$.

In Figure 9, the cutoff at a relative area of 10^{-3} is an artifact of our numerical method—it is quite difficult to reduce the error significantly. To do this one must increase the maximum number of iterations, t_{max}, to pick out narrow cha-

A pair of period three orbits is created by saddle-node bifurcation at k 5 1. At k 5 5/4 the period three saddle collides with the elliptic fixed point and there are no encircling invariant curves. Near this bifurcation, the most important feature is the virtually perfect saddle connection of the manifolds of the period three saddle ~see Fig. 10!, which has points at ~2 b,b!→~2 1 1 b,b!→~2 b,1

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[1] C. F. F. K. Karney, A. B. Rechester, and R. B. White, ‘‘Effect of noise on