

1. [2360/071423 (20 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

- (a) The equation of motion for a particular harmonic oscillator is  $x(t) = \frac{\rho\sqrt{2}}{4} \cos 2t + \frac{1}{4} \sin 2t$ . An equivalent expression is  $x(t) = \frac{\rho\sqrt{2}}{4} \cos 2t - \frac{1}{4}$ .
- (b) Any finite order, linear, homogeneous, constant coefficient differential equation that does not contain an undifferentiated term will always have a constant solution.
- (c) The differential equation  $x'' + e^x = 2$  describes a conservative system.
- (d) The function  $x(t) = e^{2t} + e^{-3t}$  is a possible solution to the differential equation for a harmonic oscillator governed by  $mx'' + bx' + kx = 0$  with mass  $m$ , damping constant  $b$  and restoring constant  $k$ .
- (e) The solution space of the differential equation  $y'''' = 0$  is span  $\{t^2 + t; t^2 + 2; 4t - 8\}$ .

**SOLUTION:**

- (a) **False**  $x(t) = \frac{\rho\sqrt{2}}{4} \cos 2t + \frac{1}{4} \sin 2t = \frac{\rho\sqrt{2}}{4} \cos 2t - \frac{1}{4}$  since  $1 = \frac{\rho\sqrt{2}}{4} \cos$  and  $1 = \frac{\rho\sqrt{2}}{4} \sin$  imply that  $\frac{\rho\sqrt{2}}{4} = \frac{1}{4}$  or  $\frac{\rho\sqrt{2}}{4} = \frac{7}{4}$ .
- (b) **True** Each term in the characteristic equation will contain an  $r$ , implying that  $r = 0$  is a root of the characteristic equation,

- (c) Rewrite the equation in the form  $y'' + \frac{5}{2t}y' + \frac{y}{2t^2} = t^2$  so that  $f(t) = t^2$  and let  $y_1 = t^{-1}$  and  $y_2 = t^{-2}$ . We assume  $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$ .

$$W[y_1; y_2](t) = \begin{vmatrix} t^{-1} & t^{-2} \\ -t^{-2} & -2t^{-3} \end{vmatrix} = \frac{1}{2}t^{-5-2} + t^{-5-2} = \frac{1}{2}t^{-5-2}$$

$$v_1(t) = \int \frac{y_2 f(t)}{W[y_1; y_2](t)} dt = \int \frac{t^{-2} t^2}{\frac{1}{2}t^{-5-2}} dt = \int 2 t^4 dt = \frac{2}{5} t^5$$

$$y_p = At + B \sin$$

$$v_2(t) = \int \frac{y_1 f(t)}{W[y_1; y_2](t)} dt = \int \frac{t^{-1} t^2}{\frac{1}{2}t^{-5-2}} dt = \int 2 t^{7-2} dt = \frac{4}{9} t^{9-2}$$

$$y_p = \frac{2}{5} t^5 t^{-1} + \frac{4}{9} t^{9-2} t^{-2} = \frac{2}{45} t^4$$

The general solution is  $y(t) = c_1 t^{-1} + c_2 t^{-2} + \frac{2}{45} t^4$ .

3. [2360/071423 (20 pts)] Characteristic equations for certain constant coefficient linear homogeneous differential equations are given, along with a forcing function,  $f(t)$ . Give the form of the particular solution you would use to solve the nonhomogeneous differential equations from which the characteristic equations were derived when using the Method of Undetermined Coefficients. **Do not** solve for the coefficients.

(a)  $r(r-2)(r-1) = 0$ ;  $f(t) = 2 + \sin t$

(b)  $[r^2 - 2r][r^2 - (2+2i)] = 0$ ;  $f(t) = \cos 2t + t$

(c)  $(r+4)(r-2) = 0$ ;  $f(t) = e^{2t} + e^{4t}$

(d)  $[r^2 - (1-i)][r^2 - (1+i)](r+3)^2 = 0$ ;  $f(t) = e^t \cos t + e^{-t} \sin t + te^{3t}$

(e)  $r^3(r-1) = 0$ ;  $f(t) = \cos 2t - \sin 3t + 1$

**SOLUTION:**

(a)  $y_p = At + B \sin t + C \cos t$

(b)  $y_p = At + B + C \cos 2t + D \sin 2t$

(c)  $y_p = Ae^{4t} + Bte^{2t}$

(d)  $y_p = te^t(A \cos t + B \sin t) + e^{-t}(C \cos t + D \sin t) + t^2(Et + F)e^{3t}$

(e)  $y_p = At^3 + B \cos 2t + C \sin 2t + D \cos 3t + E \sin 3t$

4. [2360/071423 (33 pts)] A 2-kg mass is attached to spring with restoring/spring constant of 2 Nt/m. The apparatus is aligned horizontally with a damping constant of 5 Nt/m/sec, and is forced by  $f(t) = 3e^{-t} + 4 \text{ Nt}$ . Initially,  $x(0) = -4$  and  $\dot{x}(0) = 3$ .

(a) (2 pts) Where is the mass with respect to its equilibrium position when  $t = 0$  and in what direction is it moving at that time?

(b) (3 pts) Is the oscillator over-, under-, or critically damped? Justify your answer.

(c) (3 pts) Is the oscillator in resonance? Justify your answer.

(d) (15 pts) Find the position of the mass at any time  $t$ , that is, solve an appropriate initial value problem.

(e) (15 pts) Find the amplitude of the steady-state motion.

(d) The differential equation is  $2x'' + 5x' + 2x = 3e^{-t} + 4$ . The characteristic equation for the associated homogeneous equation is

$$2r^2 + 5r + 2 = (2r + 1)(r + 2) = 0 \Rightarrow r = -\frac{1}{2}; -2 \Rightarrow x_h(t) = c_1 e^{-2t} + c_2 e^{-t/2}$$

We guess  $x_p = Ae^{-t} + B$ . Then \_