

1. [2360/092122 (35 pts)] Consider the initial value problem $(t + 1)y' - 3(t + 1)y + e^{3t} = 0$; $y(0) = \ln 3$; $t > -1$.
- (a) (4 pts) Classify _____ the equation.
 - (b) (2 pts) Does the equation possess any _____ equilibrium solutions? If so, find them.
 - (c) (7 pts)

(e) FALSE Substituting $x = 0$ and $y = 0$ into the first equation gives $x^0 = 2 \neq 0$. Going a bit further, the v-nullcline is $x^2 + y^2 = 2$ and the h-nullcline is $y^2 = x$

(b) (15 pts) Find the general solution to the differential equation.

(c) (5 pts) Solve the initial value problem.

SOLUTION :

(a) Substitute w_p into the differential equation and show that an identity results.

$$\begin{aligned}
 x \frac{dw_p}{dx} + (2x + 1)w_p &\stackrel{?}{=} 2x^2 \\
 x \left(\frac{1}{2x^2} + 1 \right) + (2x + 1) \left(\frac{1}{2x} + x \right) &\stackrel{?}{=} 2x^2 \\
 \frac{1}{2x} + x + 1 + 2x^2 &= 2x + \frac{1}{2x} + x + 1 \stackrel{?}{=} 2x^2 \\
 2x^2 &= 2x^2 \quad \checkmark
 \end{aligned}$$

(b) We need the solution, w_h , to the associated homogeneous equation.

$$\begin{aligned}
 x \frac{dw_h}{dx} + (2x + 1)w_h &= 0 \\
 \int \frac{dw_h}{w_h} &= \int \frac{2x + 1}{x} dx = \int 2 \frac{1}{x} dx \\
 \ln |w_h| &= 2x \ln |x| + c = 2x \ln x + c \quad \text{since } x > 0 \\
 |w_h| &= e^{2x \ln x + c} \\
 w_h &= \frac{C}{xe^{2x}}; \quad C \in \mathbb{R}
 \end{aligned}$$

Now apply the Nonhomogeneous Principle to obtain the general solution as

$$w = w_h + w_p = \frac{C}{xe^{2x}} + \frac{1}{2x} + x + 1$$

(c) Apply the initial condition.

$$w(1) = \frac{C}{e^2} + \frac{1}{2} + 1 + 1 = \frac{3}{2} \Rightarrow C = e^2$$

giving the solution to the initial value problem as

$$w(x) = \frac{e^{2-2x}}{x} + \frac{1}{2x} + x + 1$$