1. (32 pts) Suppose the density of the surface $z = 1 - x^2$ is $\delta = |xy|$ g/cm^2 and consider the vector field

$$\mathbf{F} = \langle 3x + \cos y, 2y + \sin z, e^x + 5z \rangle$$

   (a) Find the mass of the part of the surface lying above the region in the xy-plane between $y = 0$ and $y = 2$.

   (b) Find the outward flux of $\mathbf{F}$ through the **closed** surface enclosing the region below $z = 1 - x^2$, above the xy-plane and between $y = 0$ and $y = 2$.

Solution:

(a)

$$g(x, y, z) = x^2 + z \implies \nabla g = \langle 2x, 0, 1 \rangle \implies \|\nabla g\| = \sqrt{4x^2 + 1}$$

Project surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$, integration region $-1 \le x \le 1, 0 \le y \le 2$ and $|\nabla g \cdot \mathbf{p}| = 1$

$$\begin{aligned}
\text{Mass} &= \int_{-1}^{1}\int_{0}^{2} |xy|\sqrt{4x^2+1}\,dy\,dx \quad \text{(integrand even in } x \text{ and separable)} \\
&= 2\int_0^1 x\sqrt{4x^2+1}\,dx \int_0^2 y\,dy \\
&= \frac{1}{4}\int_1^5 u^{1/2}\,du \left[\frac{y^2}{2}\right]_0^2 = \frac{1}{3}\left(5\sqrt{5} - 1\right)
\end{aligned}$$

(b) The surface S and the region W it encloses satisfy the hypotheses of Gauss' (Divergence) Theorem with

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(3x + \cos y) + \frac{\partial}{\partial y}(2y + \sin z) + \frac{\partial}{\partial z}(e^x + 5z) = 3 + 2 + 5 = 10$$

$$\begin{aligned}
\text{Flux} &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F}\,dV = \int_{-1}^{1}\int_0^2\int_0^{1-x^2} 10\,dz\,dy\,dx \\
&= 10\int_{-1}^{1}\int_0^2 \left(1 - x^2\right)dy\,dx = 20\int_{-1}^{1}\left(1 - x^2\right)dx = 40\int_0^1 \left(1 - x^2\right)dx = \frac{80}{3}
\end{aligned}$$

2. (16 pts) Find the area under the graph of $z = 100(x^2 + 2y^2)$ lying above the second quadrant portion of the **curve** $x^2 + y^2 = 4$.

Solution: The area is given by $\displaystyle\int_C f(x, y)\,ds$ where $f(x, y) = 100(x^2 + 2y^2)$. C can be parameterized by

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad \pi/2 \le t \le \pi \implies \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} \implies \|\mathbf{r}'(t)\| = 2$$

Thus

$$\begin{aligned}
\text{Area} &= \int_C 100(x^2 + 2y^2)\,ds = \int_{\pi/2}^{\pi} 100\left(4\cos^2 t + 8\sin^2 t\right)(2)\,dt = 800\int_{\pi/2}^{\pi}\left(\cos^2 t + 2\sin^2 t\right)dt \\
&= 800\int_{\pi/2}^{\pi}\left(1 + \sin^2 t\right)dt = 800\int_{\pi/2}^{\pi}\left(\frac{3}{2} - \frac{1}{2}\cos 2t\right)dt = 400\left[3t + \frac{1}{2}\sin 2t\right]_{\pi/2}^{\pi} = 600\pi
\end{aligned}$$

3. (16 pts) I am doing laps around the unit circle (counterclockwise) in the presence of the force field

$$\mathbf{F} = \left\langle Axy - By^3, 4y + 3x^2 - 3xy^2 \right\rangle$$

(a) After having gone from $(1, 0)$ to $(0, 1)$, I am already getting tired from all of the work I've done. A friend standing nearby tells me to chill because when I get back to $(1, 0)$ I will have done no work at all. What are A and B? Briefly explain.

(b) If I go around the circle too much, I'll get dizzy so my friend tells me to go from $(-1, 0)$ to $(3, -2)$ along the path $y = \sqrt{x+1}(x-2)^{300}(x-4)^{301}$ instead. How much work will I do walking on that path?

Solution:

(a) The fact that no work is done when traversing a closed path implies that the vector field is conservative so that

$$\frac{\partial}{\partial x}\left(4y + 3x^2 - 3xy^2\right) = \frac{\partial}{\partial y}\left(Axy - By^3\right) \implies 6x - 3y^2 = Ax - 3By^2 \implies A = 6, B = 1$$

(b) Parameterizing the given path would not be a pleasant experience but that is not necessary. There are two ways to handle this. First, since the vector field is conservative, line integrals are path independent so we could pick another path between the given points (perhaps a line segment). This may still be too much work (no pun intended). The other approach is to find a potential function and use the fundamental theorem for line integrals to compute the work. To that end,

$$\frac{\partial f}{\partial x} = 6xy - y^3 \implies f(x, y) = \int 6xy - y^3 \, dx = 3x^2y - xy^3$$

(This was easily obtained since we knew where the plane intersects the coordinate axes. A point and two vectors in the plane could also have been used to find the plane's equation).

To obtain the orientation of the surface induced by the orientation of its boundary requires the use of ∇g. Projecting the surface onto the xy-plane gives $\mathbf{p} = \mathbf{k}$ and $|\nabla g \cdot \mathbf{p}| = 1$ with the area of integration

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1; 0 \le y \le 2 - 2x\}$$

Note that the surface could have been projected onto the xz- or yz-plane.

We need the curl of $\mathbf{V}$, given as

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & x + yz & xy - \cos^2 \sqrt{z} \end{vmatrix} = (x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$$

Then

$$\begin{aligned} \text{Circulation} = \oint_C \mathbf{V} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{V} \cdot d\mathbf{S} = \iint_R (\nabla \times \mathbf{V}) \cdot \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|}\, dA \\ &= \iint_R [(x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}] \cdot \frac{(-2\mathbf{i} - \mathbf{j} - \mathbf{k})}{1}\, dA \\ &= \int_0^1 \int_0^{-2x+2} (-2x + 3y - 1)\, d \end{aligned}$$

END OF EXAM