1 Evaluate the following integrals. Be sure to simplify your answers.

(a) (14 points) $\int_2^2 \frac{1}{x^2} dx$

(b) (14 points) $\int t^5 \sin(t^3) dt$

Solution: (a) Since there is a discontinuity in the integrand at $x = 0$, this is an improper integral that we have to split into two integrals, i.e.,

$$\begin{aligned}
\int_2^2 \frac{1}{x^2}\,dx &= \lim_{s \to 0} \int_2^s \frac{1}{x^2}\,dx + \lim_{t \to 0^+} \int_t^2 \frac{1}{x^2}\,dx \\
&= \lim_{s \to 0} \left[-\frac{1}{x} \right]_2^s + \lim_{t \to 0^+} \left[-\frac{1}{x} \right]_t^2 \\
&= \lim_{s \to 0} \left(-\frac{1}{s} - \frac{1}{2} \right) + \lim_{t \to 0^+} \left(-\frac{1}{2} + \frac{1}{t} \right) \\
&= \lim_{s \to 0} -\frac{1}{s} - 1 + \lim_{t \to 0^+} \frac{1}{t};
\end{aligned}$$

Since both limits diverge to positive in nity, the improper integral also diverges to positive in nity.

(b) We perform integration by parts for this problem. In order to do so, we rst perform a u-substitution, taking $u = t^3$ and $du = 3t^2\,dt$ (this can be rearranged to be obtain $dt = \frac{du}{3t^2}$). The integral then becomes

$$\begin{aligned}
\int t^5 \sin\left(t^3\right)\,dt &= \frac{1}{3} \int u \sin(u) \frac{t^2}{t^2}\,du \\
&= \frac{1}{3} \int u \sin(u
\end{aligned}$$

(a) (8 points) Solve explicitly for for the general solution $y(x)$.

(b) (8 points) From the general solution $y(x)$ derived in part (a), solve for the constant of integration c.

Solution: (a) This is a separable ODE, which we can rearrange the ODE as follows:

$$\frac{dy}{dx} = \frac{x^2}{y} \implies y\,dy = x^2\,dx.$$

We now integrate both sides of the rearranged ODE:

$$\int y\,dy = \int x^2\,dx \implies \frac{y^2}{2} + c = \frac{x^3}{3}$$
$$\implies y^2 = \frac{2}{3}x^3 + c$$
$$\implies y(x) = \pm\sqrt{\frac{2}{3}x^3 + c}.$$

Since we take the square root of both sides, we have to consider both the negative and positive square root of the right-hand side.

(b) To determine the constant of integration c in the general solution from earlier, we impose the given initial condition. Since $y(0) = -5$, this implies that we take the negative square root of the right-hand side from earlier, i.e., we take $y(x) = -\sqrt{\frac{2}{3}x^3 + C}$. Imposing the initial condition on y, we have that

$$-5 = -\sqrt{\frac{2}{3}(0)^3 + C} \implies 25 = C.$$

Thus, the solution that satisfies the above ODE and the given initial condition is $y(x) = -\sqrt{\frac{2}{3}x^3 + 25}$.

3 (10 points) Consider the curve defined by $y = \sec(x)$ on $0 \le x \le \frac{\pi}{4}$. Set up but **do not evaluate** the surface area of the solid obtained by rotating the curve about the y-axis.

Solution: Since we are rotating the above curve about the y-axis, we use the following surface area of a solid of revolution formula:

$$SA = 2\pi \int_a^b r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

Here, r is the distance from the axis of rotation to the curve, which is simply x in this case. The lower and upper bounds of integration are obtained by taking the lower and upper bounds on x that are provided, i.e.,

$$a = 0;$$

$$b = \frac{\pi}{4};$$

(b) $\int f(x)dx = \int \frac{1}{1+x^2}dx = \tan^{1}($