

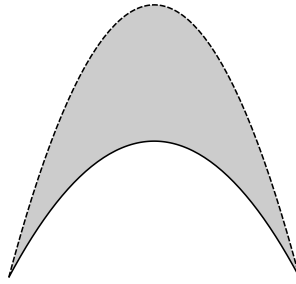
1. (22 points) Consider the region in the first quadrant bounded by $y = \sin x$ and $y = \frac{2x}{\pi}$. Set up but do not evaluate the integrals to find the following quantities:
- Graph the given equations and shade the region. Label the equations and intersection points.
 - The volume of a solid with a base given and cross-sections perpendicular to the x -axis that are isosceles triangles with a height equal to the length of the base. (The base of the isosceles triangles is in the xy -plane.)
 - The volume generated by rotating about the line $x = 3$.
 - The perimeter of the region.

Solution:

- (a) Graphing the region and labeling all of the important features gives us



- (a) (10 points) Find the center of mass of the region bounded by $y = 2(1 - x^2)$ for $y > 0$. Assume a constant density. The region is shown below.



- (b) (8 points) Let R be the radius of the Earth. The gravitational force on a mass m at a height x above the Earth's surface has magnitude $F(x) = \frac{mgR^2}{(R + x)^2}$. How much work is required to move the mass from a height $x = 0$ to a height $x = H$? (Assume $H > 0$. R , m , and g are fixed constants.)

- (c) (12 points) Find the solution of the differential equation $\frac{dy}{dx} = \ln(x)$ with initial condition $y(1) = e$. Express your answer in the form $f(x)$.

Solution:

- (a) By symmetry the moment about the y -axis is zero so is also zero. To find we only need to find the moment about the x -axis and the total mass. The total mass is

$$M = \int_{-1}^1 2(1 - x^2) \int_0^{2(1-x^2)} (1 - x^2) dy dx = \int_{-1}^1 (1 - x^2) dx$$

Using symmetry we could also write

$$M = 2 \int_0^1 (1 - x^2) dx:$$

The total mass is

$$M = 2 \int_0^1 (1 - x^2) dx = 2 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{4}{3}:$$

The moment about the x -axis is

$$M_x = \int_{-1}^1 2(1 - x^2) \int_0^{2(1-x^2)} (1 - x^2)^2 dy dx = \int_{-1}^1 (1 - x^2)^2 dx = \int_{-1}^1 (1 - 2x^2 + x^4) dx = \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^1 = \frac{8}{5}:$$

Finally, the coordinates of the center of mass are

$$\boxed{x = 0; y = \frac{6}{5}}:$$

- (b) The total work is obtained by the integral

$$W = \int_0^H F(x) dx = \int_0^H \frac{mgR^2}{(R + x)^2} dx$$

Make the substitution

$$u = R + x; \quad du = dx$$

which produces the integral

$$W = \int_R^{R+H} \frac{mgR^2}{u^2} du = \frac{mgR^2}{u} \Big|_{u=R}^{u=R+H} = \boxed{mgR \left(\frac{mgR^2}{R + H} = \frac{mgHR}{R + H} \right)}:$$

(c)

Solution:

(a) We note that $a_n = 1 + \frac{\ln 2}{n} = e^{\ln(1 + \frac{\ln 2}{n})} = e^{\frac{\ln(1 + \frac{\ln 2}{n})}{1 + \frac{\ln 2}{n}}}$. Working with the part in the exponent, we have

$$\lim_{n \rightarrow \infty} \frac{\ln(1 + \frac{\ln 2}{n})}{1 + \frac{\ln 2}{n}} \stackrel{L'H}{=} \lim_{n \rightarrow \infty} \frac{\frac{1}{1 + \frac{\ln 2}{n}} \cdot \frac{\ln 2}{n^2}}{\frac{-1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\ln 2}{1 + \frac{\ln 2}{n}} = \ln 2$$

Putting this result into the original sequence gives

$$\lim_{n \rightarrow \infty} a_n = e^{\ln 2} = \boxed{2}$$

(b) The sequence converges to $\lim_{n \rightarrow \infty} 4^{2+3n} = \lim_{n \rightarrow \infty} (4^{2+3n})^{1/n} = \lim_{n \rightarrow \infty} 4^{(2/n)+3} = \boxed{64}$

(c) The series $\sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n}{3^n}$ is the sum of two convergent geometric series:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2^{n+1} + (-1)^n}{3^n} &= \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} + \sum_{n=0}^{\infty} \frac{1}{3^n} \\ &= \frac{1}{2} \frac{1}{1 - \frac{2}{3}} + \frac{1}{1 - \frac{1}{3}} \\ &= \frac{3}{2} + \frac{3}{4} = \boxed{\frac{9}{4}} \end{aligned}$$

(d) Using the divergence test, we see:

$$\lim_{n \rightarrow \infty} \frac{n}{n}$$

(b) Before we write out a simplified, general expression for a_n , we can use partial fractions to write

$$a_n = \frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Then, the sum of the first n terms of the series (including enough terms to see a pattern of cancellation) is given by

$$\begin{aligned} s_n &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2} \\ &= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \\ &= \boxed{\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}}. \end{aligned}$$

(c) To find the sum of the series, we just need to take the limit of the partial sums found in part (b). This yields the convergent series

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \boxed{\frac{3}{2}}.$$