(b) i.

$$\int x^2 \ln x\, dx = \int \underbrace{x^2}_{\substack{v = x^3/3 \\ dv = x^2\,dx}} \underbrace{\ln x}_{\substack{u=\ln x \\ du = dx/x}} dx \overset{IBP}{=} \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx$$

$$= \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + C}$$

ii.

$$\int_0^1 x^2 \ln x\, dx = \lim_{t \to 0^+} \int_t^1 x^2 \ln x\, dx$$

$$= \lim_{t \to 0^+} \left[\frac{x^3}{3}\ln x - \frac{x^3}{9} \right]_t^1$$

$$= \lim_{t \to 0^+} \left(0 - \frac{1}{9} - \frac{t^3}{3}\ln t + \frac{t^3}{9} \right) = \boxed{-\frac{1}{9}}$$

because $\lim_{t \to 0^+} t^3 \ln t = \lim_{t \to 0^+} \frac{\ln t}{t^{-3}} \overset{LH}{=} \lim_{t \to 0^+} \frac{t^{-1}}{-3t^{-4}} = \lim_{t \to 0^+} -\frac{t^3}{3} = 0.$

3. (22 pts) Find the value the sequence or series converges to. If it does not converge, explain why not.

(a) $\left\{ \frac{\sqrt{4n}}{1+\sqrt{n}} \right\}$

(b) $\sum_{n=1}^{\infty} \frac{1}{3n+2}$

(c) $\sum_{n=1}^{\infty} \frac{5-2^n}{5^n}$

Solution:

(a) $\lim_{n \to \infty} \frac{\sqrt{4n}}{1+\sqrt{n}} = \lim_{n \to \infty} \frac{2\sqrt{n}}{1+\sqrt{n}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}$

4. (15 pts) Let $f(x) = x \ln x - x + 1$.

(a) Use the formula for Taylor Series to find the polynomial $T_2(x)$ for $f(x)$ centered at $a = 1$.

(b) Suppose $T_2(x)$ is used to approximate $f\left(\frac{3}{2}\right)$. By the Alternating Series Estimation Theorem, what is an error bound for the approximation? Note: The series corresponding to $f\left(\frac{3}{2}\right)$ is alternating and satisfies the conditions of the theorem.

Solution:

(a) The Taylor Series for a function $f(x)$ centered at 1 is $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n.$

The first two derivatives of $f(x) = x \ln x - x + 1$ are

$$f'(x) = 1 + \ln x - 1 \qquad f'(1) = 0$$
$$f''(x) = \frac{1}{x} \qquad f''(1) = 1.$$

It follows that

$$T_2(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2$$
$$= 0 + 0 + \frac{1}{2!}(x-1)^2 = \boxed{\frac{1}{2}(x-1)^2}.$$

(b) The series centered at 1 corresponding to $f(3/2)$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}\left(\frac{1}{2}\right)^n.$

The approximation $T_2(3/2)$ equals the sum of the first 3 terms of the series. By the Alternating Series Estimation Theorem, an error bound is the magnitude of the next term:

$$\left|\frac{f^3(1)}{3!}\left(\frac{1}{2}\right)^3\right|.$$

The third derivative of f is $f''' = -1/x^2$ and $f'''(1) = -1$, so an error bound is

$$\left|\frac{f^3(1)}{3!}\left(\frac{1}{2}\right)^3\right| = \left|-\frac{1}{3!}\left(\frac{1}{2}\right)^3\right| = \boxed{\frac{1}{48}}.$$

5. (20 pts) Let $g(x) = \arctan\left(x^2\right)$.

(a) Find a Maclaurin series for $g(x)$.

(b) Use your answer for part (a) to find a Maclaurin series for $x^3 g'(x)$. Simplify your answer.

(c) What is the sum of the series found in part (b)?

Solution:

(a) The Maclaurin series for $\arctan x$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

The Maclaurin series for $g(x) = \arctan\left(x^2\right)$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{2n+1}$sIJ/F3410.9091Tf4Td65J/F4810.9091Tf7.72

(b) $x^3 g'(x) = x^3 \dfrac{d}{dx}\left(\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{4n+2}}{2n+1}\right) = x^3 \sum_{n=0}^{\infty} (-1)^n \dfrac{(4n+2)x^{4n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty} (-1)^n 2x^{4n+4}}$.

(c) The sum of the series is $x^3 g'(x) = x^3 \dfrac{d}{dx}\left(\arctan\left(x^2\right)\right) = x^3 \left(\dfrac{2x}{1+x^4}\right) = \boxed{\dfrac{2x^4}{1+x^4}}$.

Alternate solution:

The series $\sum_{n=0}^{\infty} (-1)^n 2x^{4n+4} = \sum_{n=0}^{\infty} \underbrace{2x^4}_{a} \underbrace{\left(-x^4\right)^n}_{r^n}$ is geometric with first term $a = 2x^4$ and ratio $r = -x^4$. The sum of the series is therefore $S = \dfrac{a}{1-r} = \dfrac{2x^4}{1+x^4}$.

6. (14 pts) Consider the parametric curve $x = e^{t/2}$, $y = 1 + e^{2t}$.

(a) Find an equation of the line with slope 4 that is tangent to the curve.

(b) Eliminate the parameter to find a Cartesian equation of the curve. Simplify your answer.

Solution:

(a) The slope of the curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2e$$

(b) Apply the identities $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} x^2 &= 16 + 16y^2 \\ r^2\cos^2\theta &= 16 + 16r^2\sin^2\theta \\ r^2\cos^2\theta - 16r^2\sin^2\theta &= 16 \\ r^2 &= \frac{16}{\cos^2\theta - 16\sin^2\theta} \\ r &= \boxed{\sqrt{\frac{16}{\cos^2\theta - 16\sin^2\theta}}} \end{aligned}$$

8. (20 pts) Consider the polar curves $r = 2 + \sin(2\theta)$ and $r = 2 + \cos(2\theta)$ in the 1st and 2nd quadrants, shown at right.

   (a) Find the (x, y) coordinates for the point that corresponds to $r = 2 + \sin(2\theta)$, $\theta = \frac{\pi}{6}$. Simplify your answer.

   (b) Set up (but <u>do not evaluate</u>) integrals to find the following quantities.

      i. Length of the curve $r = 2 + \sin(2\theta)$.

      ii. Area of the region inside $r = 2 + \sin(2\theta)$ and outside $r = 2 + \cos(2\theta)$. *Hint:* For the bounds, consider $\tan(2\theta)$.

Solution:

(a) $x = r\cos\theta = (2 + \sin(\pi/3))\cos(\pi/6) = \left(2 + \frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{2} = \boxed{\sqrt{3} + \frac{3}{4}}$

$y = r\sin\theta = (2 + \sin(\pi/3))\sin(\pi/6) = \left(2 + \frac{\sqrt{3}}{2}\right)\frac{1}{2} = \boxed{1 + \frac{\sqrt{3}}{4}}$

(b) i. $L = \displaystyle\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \boxed{\int_0^{\pi} \sqrt{(2 + \sin(2\theta))^2 + (2\cos(2\theta))^2}\, d\theta}$

ii. First find the intersection points.

$$\begin{aligned} 2 + \sin(2\theta) &= 2 + \cos(2\theta) \\ \sin(2\theta) &= \cos(2\theta) \\ \tan(2\theta) &= 1 \\ 2\theta &= \frac{\pi}{4}, \frac{5\pi}{4} \\ \theta &= \frac{\pi}{8}, \frac{5\pi}{8} \end{aligned}$$

The area between the curves is

$$A = \int \frac{1}{2}\left(r_1^2 - r_2^2\right) d\theta = \boxed{\int_{\pi/8}^{5\pi/8} \frac{1}{2}\left((2 + \sin(2\theta))^2 - (2 + \cos(2\theta))^2\right) d\theta}$$