Name: ______________________________

APPM 1350
Summer 2024

Exam 2

June 28

Instructions:

- Write your name at the top of each page.
- Show all work and simplify your answers, except where the instructions tell you to leave your answer unsimplified.
- Be sure that your work is legible and organized.
- Name any theorem that you use and explain how it is used.
- Answers with no justification will receive no points unless the problem explicitly states otherwise.
- Notes, your text and other books, calculators, cell phones, and other electronic devices are not permitted, except as needed to upload your work.
- When you have completed the exam, upload it to Gradescope. Verify that everything has been uploaded correctly and pages have been associated to the correct problem before you leave the room.
- Turn in your hardcopy exam before you leave the room.

Half / Double Angle Formulas

- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 1 - 2\sin^2(\theta) \\ -1 + 2\cos^2(\theta) \end{cases}$
- $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1-\cos(\theta))}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1}{2}(1+\cos(\theta))}$
- $\tan\left(\frac{\theta}{2}\right) = \begin{cases} \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}} \\ \frac{\sin(\theta)}{1+\cos(\theta)} \\ \frac{1-\cos(\theta)}{\sin(\theta)} \end{cases}$

Angle Sum / Difference Formulas

- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

1. (40 pts) The following problems are not related.

   (a) Find $\frac{dy}{dx}$ if $x\cos(y) + y^3 = {}^{3/2}$. You should solve for $\frac{dy}{dx}$, but you do not need to simplify your answer further.

   (b) Use linearization to estimate $(8.01)^{2/3}$. You do not need to simplify your answer.

   (c) Show that $f(x) = x^2 - 3x - 5$ satisfies the three hypotheses of Rolle's Theorem on the interval $[-2, 5]$. For what value(s) of c is the conclusion satisfied?

   (d) Find the absolute maximum and minimum values of $f(x) = \dfrac{x^2}{x+2}$ on the interval $[-6, -5/2]$.

Solution:

(a)

$$\begin{aligned}\frac{d}{dx}\quad x\cos(y) + y^3 &= {}^{3/2}\\ \cos(y) - x\sin(y)\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 0\\ \frac{dy}{dx}\quad 3y^2 - x\sin(y) &= -\cos(y)\\ \frac{dy}{dx} &= -\frac{\cos(y)}{3y^2 - x\sin(y)}\end{aligned}$$

(b) One option is to linearize the function $f(x) = x^{2/3}$ at $a = 8$. Finding the slope of the tangent line:

$$f'(x) = \frac{2}{3x^{1/3}} \tag{1}$$

$$f'(8) = \frac{2}{3(8)^{1/3}} = \frac{2}{3\cdot 2} = \frac{1}{3} \tag{2}$$

And $f(8) = 8^{2/3} = 2^2 = 4$. So the linearization of f at $a = 8$ is

$$L(x) = 4 + \frac{1}{3}(x - 8)$$

and $f(8.01) = (8.01)^{2/3}$ is approximated by

$$\begin{aligned}f(8.01) \approx L(8.01) &= 4 + \frac{1}{3}(0.01)\\ &= 4 + 0.3\overline{3}(0.01)\\ &= 4.003\overline{3}\end{aligned}$$

(c) Since f is a polynomial, it is continuous everywhere. $f'(x) = 2x - 3$ is also a polynomial and continuous everywhere. Therefore f satisfies the first two conditions. Evaluating at the endpoints, we have

$$\begin{aligned}f(-2) &= 4 + 6 - 5 = 5\\ f(5) &= 25 - 15 - 5 = 5\end{aligned}$$

so $f(-2) = f(5)$ and f satisfies the third condition. The conclusion of Rolle's Theorem is that there must be a c in $(-2, 5)$ such that $f'(c) = 0$. Since

$$f'(c) = 2c - 3 = 0 \quad (\ ) \quad c = \frac{3}{2}$$

the value of c that satisfies the conclusion is $c = 3/2$.

(d) The derivative is:

$$f'(x) = \frac{x^2 + 4x}{(x + 2)^2}$$

Therefore there are critical numbers for which $f'(x) = 0$ at $x = 0, -4$. The critical number $x = -4$ is within the given interval. Testing the critical number and endpoints,

$$f(-6) = \frac{36}{-4} = -9$$

$$f(-4) = \frac{16}{-2} = -8$$

$$f(-5/2) = \frac{25/4}{-1/2} = -\frac{25(2)}{4} = -\frac{25}{2} = -12.5$$

Therefore the absolute minimum is at $(-5/2, -25/2)$ and the absolute maximum is at $(-$

2. (16 pts) A small ship carrying nothing but rubber ducks hits an iceberg and begins to spill its cargo into the ocean. The rubber-duck spill is circular and the radius increases at a rate of $0{:}5$m=s. At what rate is the area of the spill increasing when the radius is 10m?

Solution: Let r be the radius of the spill and A be the area. We know that $\mathrm{d}r{=}\mathrm{d}t = 0{:}5$m=s. Thus

$$\begin{aligned} A &= \ r^2 \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= 2\ r\frac{\mathrm{d}r}{\mathrm{d}t} \\ \frac{\mathrm{d}A}{\mathrm{d}t}\bigg|_{r=10} &= 2\ (10)(0{:}5) \\ &= 10\ \ \mathrm{m}^2{=}\mathrm{s} \end{aligned}$$

3. (44 pts) Let $f(x) = \dfrac{x^2 - 9}{x + 5}$. The first and second derivatives of f are:

$$f'(x) = \frac{x^2 + 10x + 9}{(x+5)^2}$$
$$f''(x) = \frac{32}{(x+5)^3}$$

(a) Find the domain of f.

(b) Find all x- and y-intercepts of f.

(c) Find all vertical, horizontal, and slant asymptotes of f.

(d) Find the intervals over which f is increasing and decreasing.

(e) Find and classify all local extrema.

(f) Find the intervals over which f is concave up and concave down. List points of inflection, if any.

(g) Sketch f. Be sure to label all relevant quantities. You may use the blank graph provided on the next page if you wish.

Solution:

(a) The denominator is zero when $x = -5$, and there are no other restrictions on the domain. Therefore the domain is $(-\infty, -5) \cup (-5, \infty)$

(b) By solving $0 = x^2 - 9 = (x+3)(x-3)$ we see that there are x-intercepts at $(-3, 0)$ and $(3, 0)$. There is a y-intercept at $(0, -9/5)$.

(c) There is a potential vertical asymptote at $x = -5$. We can prove this using at least one of the following one-sided limits:

$$\lim_{x \to -5^+} \frac{x^2 - 9}{x + 5} = \frac{16}{0^+} = \infty$$

or

$$\lim_{x \to -5^-} \frac{x^2 - 9}{x + 5} = \frac{16}{0^-} = -\infty$$

Therefore there is a vertical asymptote at $x = -5$. Since the degree of the numerator is one more than the denominator, there is also a slant asymptote. We can find it using polynomial long division.

$$\begin{array}{r|l} & x \quad - 5 \\ \hline x + 5 & x^2 \qquad\quad - 9 \\ & -x^2 - 5x \\ \hline & \qquad -5x \quad - 9 \\ & \qquad\ \ 5x + 25 \\ \hline & \qquad\qquad\ \ 16 \end{array}$$

So $\frac{x^2-9}{x+5} = x - 5 + \frac{16}{x+5}$, and the slant asymptote is at $y = x - 5$. To prove this with limits, you could write

$$\lim_{x \to \infty} f(x) - (x - 5) = \lim_{x \to \infty} \frac{16}{x+5} = 0.$$

(d) Finding the critical numbers:

$$x^2 + 10x + 9 = 0$$
$$(x + 9)(x + 1) = 0$$

There are critical points for which $f(x) = 0$ at $x = -9, -1$ and a critical point for which $f'(x)$ DNE at $x = -5$. Testing points,

$$f'(-10) = \frac{100 - 100 + 9}{25} > 0$$
$$f'(-6) = \frac{36 - 60 + 9}{1} < 0$$
$$f'(-4) = \frac{16 - 40 + 9}{1} < 0$$
$$f'(0) = 9/5 > 0$$

The function is increasing on $(-\infty, -9) \cup (-1, \infty)$ and decreasing on $(-9, -5) \cup (-5, -1)$.

(e) Finding the y values at the critical points where $f'(x) = 0$,

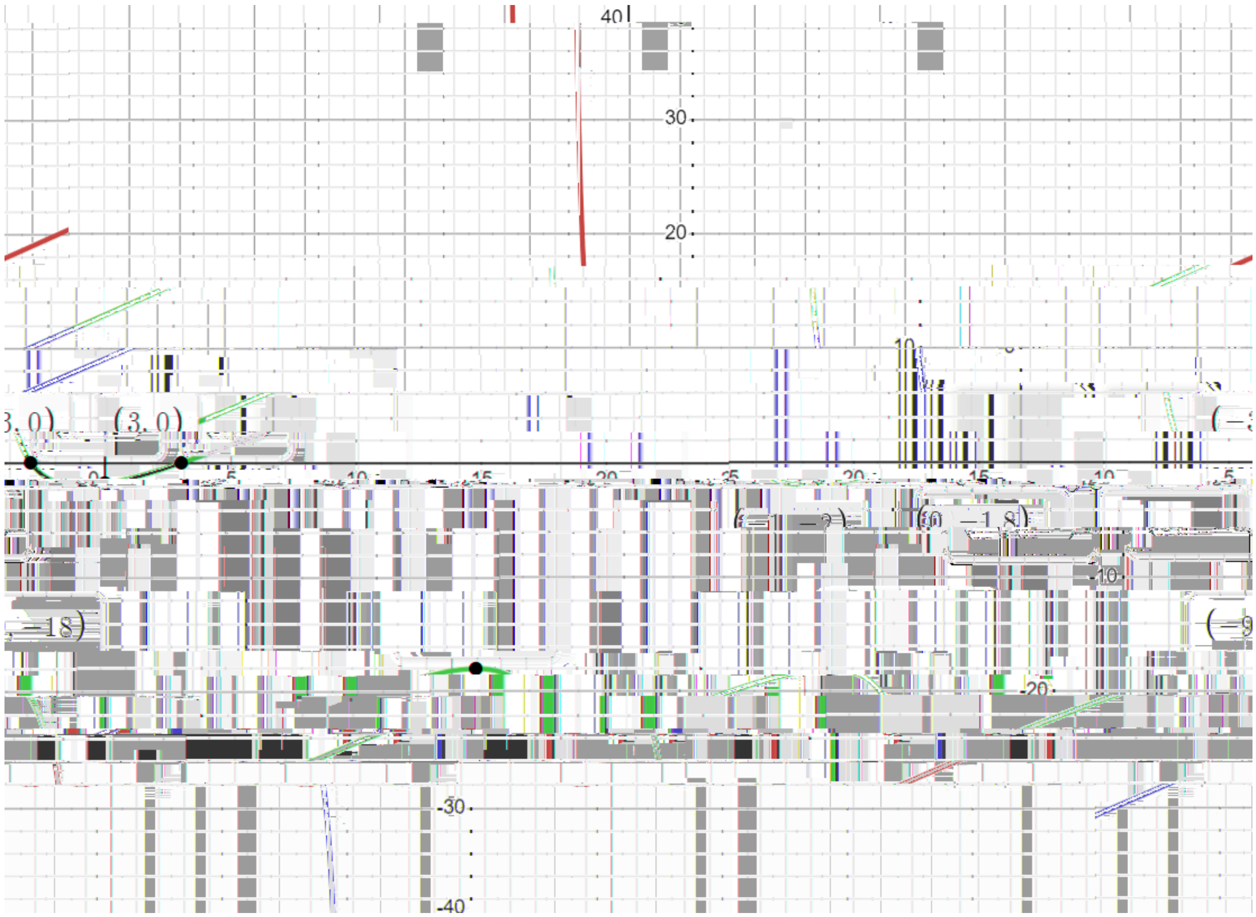
$$f(-9) = \frac{81 - 9}{4} = \frac{72}{4} = 18$$
$$f(-1) = \frac{1 - 9}{4} = -2$$

We can classify the points using the first derivative test as performed in part (d). There is therefore a local maximum at $(-9, -18)$ and a local minimum at $(-1, -2)$.

(f) There are no x-values for which $f''(x) = 0$. The one point where $f''(x)$ DNE is at $x = -5$. Testing points,

$$f''(-6) = \frac{32}{(-1)^3} < 0$$
$$f''(-4) = \frac{32}{(1)^3} > 0$$

Therefore f is concave down on $(-\infty, -5)$ and concave up on $(-5, \infty)$. There are no points of inflection. (



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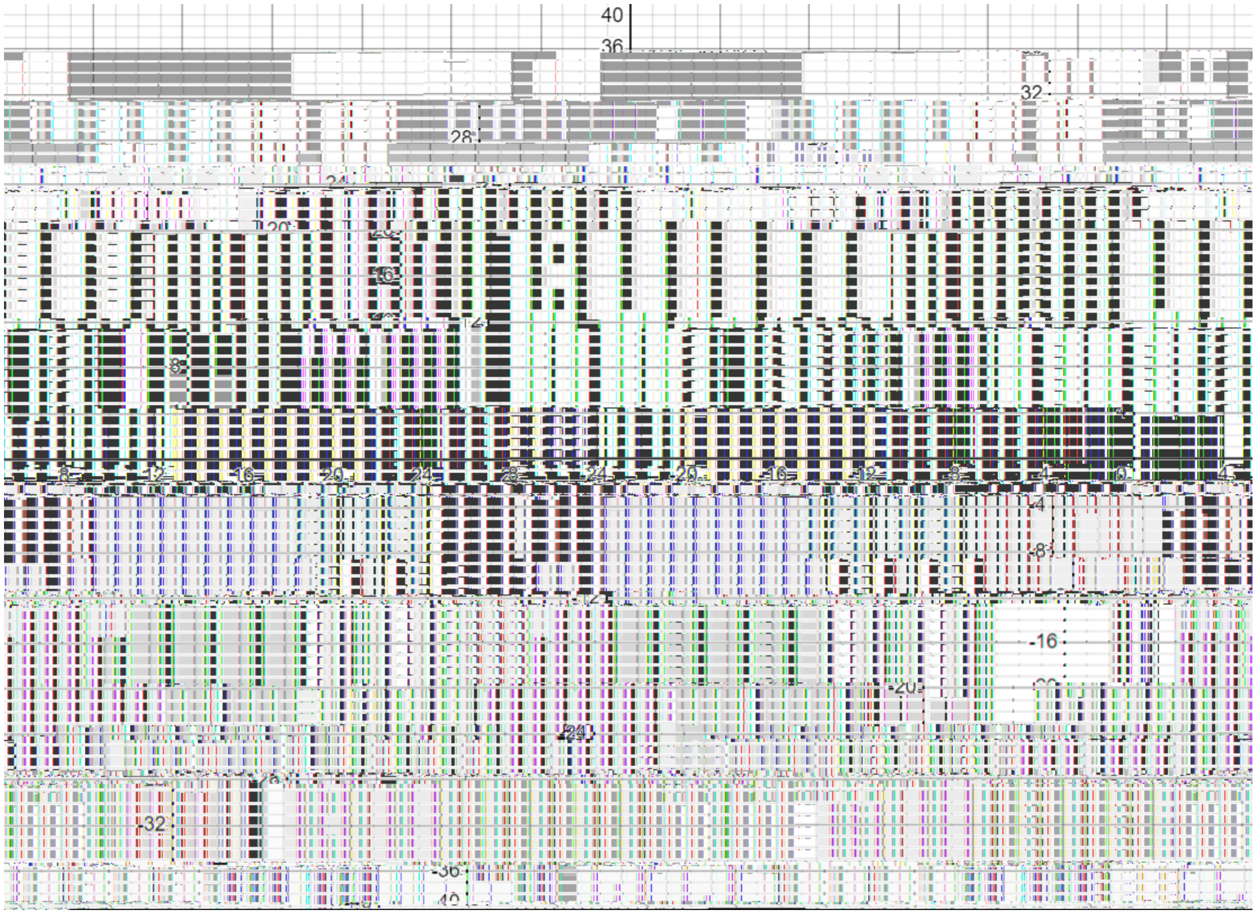

THIS IS THE END OF THE EXAM