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Abstract

A discrete-time dynamical system is considered. The system is defined by a map  $T$  on a space  $X$ . The map  $T$  is assumed to be a contraction mapping. The fixed point of  $T$  is denoted by  $x^*$ . The convergence of the iterates of  $T$  to  $x^*$  is studied. The convergence is shown to be exponential. The rate of convergence is determined by the spectral radius of the derivative of  $T$  at  $x^*$ . The convergence is also studied in the case where  $T$  is not a contraction mapping. The convergence is shown to be polynomial. The rate of convergence is determined by the order of the first non-linear term in the Taylor expansion of  $T$  at  $x^*$ .

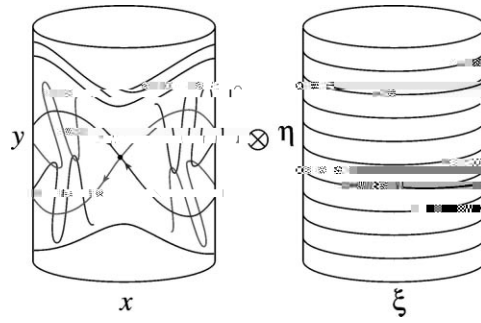
MSC: 37J40; 37L05; 37N05

Keywords: Discrete-time dynamical system; Contraction mapping; Fixed point; Exponential convergence; Polynomial convergence

1. Introduction

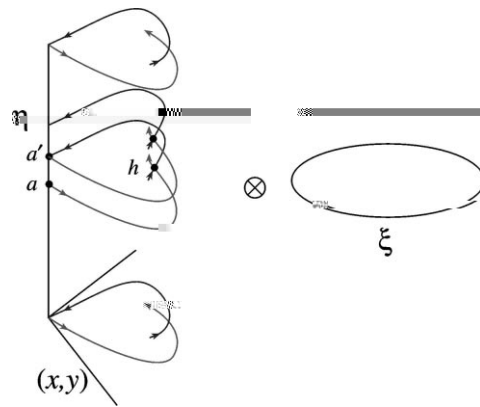
A discrete-time dynamical system is considered. The system is defined by a map  $T$  on a space  $X$ . The map  $T$  is assumed to be a contraction mapping. The fixed point of  $T$  is denoted by  $x^*$ . The convergence of the iterates of  $T$  to  $x^*$  is studied. The convergence is shown to be exponential. The rate of convergence is determined by the spectral radius of the derivative of  $T$  at  $x^*$ . The convergence is also studied in the case where  $T$  is not a contraction mapping. The convergence is shown to be polynomial. The rate of convergence is determined by the order of the first non-linear term in the Taylor expansion of  $T$  at  $x^*$ .

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... (1)  $w = h = 0$  ...  $(x, y)$  ...

...  $T^2 \times R^2$  ...  $x' = x + y'$ ,  $y' = y - k(1 + h \dots) x$ ,  $\dots = -kh(\dots)$



2.  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are the components of  $\mathcal{C}$  that contain  $a$  and  $a'$  respectively. If  $h \neq 0$ , then  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are disjoint. If  $h = 0$ , then  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are adjacent.

Let  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  be the components of  $\mathcal{C}$  that contain  $a$  and  $a'$  respectively. If  $h \neq 0$ , then  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are disjoint. If  $h = 0$ , then  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are adjacent. Let  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  be the components of  $\mathcal{C}$  that contain  $a$  and  $a'$  respectively. If  $h \neq 0$ , then  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are disjoint. If  $h = 0$ , then  $\mathcal{C}_a$  and  $\mathcal{C}_{a'}$  are adjacent.



$(x, y) \mapsto (x', y'),$   
 $x' = x + \frac{1}{2}y', \quad y' = y + V(x).$ 
(4)

$n = 1, \dots, V(x) = k(x),$

**3. A t - t ab - t**

$\mathbf{A} = \dots, \mathbf{B} = \dots, \mathbf{C} = \dots, \mathbf{D} = \dots, \mathbf{E} = \dots, \mathbf{F} = \dots, \mathbf{G} = \dots, \mathbf{H} = \dots, \mathbf{I} = \dots, \mathbf{J} = \dots, \mathbf{K} = \dots, \mathbf{L} = \dots, \mathbf{M} = \dots, \mathbf{N} = \dots, \mathbf{O} = \dots, \mathbf{P} = \dots, \mathbf{Q} = \dots, \mathbf{R} = \dots, \mathbf{S} = \dots, \mathbf{T} = \dots, \mathbf{U} = \dots, \mathbf{V} = \dots, \mathbf{W} = \dots, \mathbf{X} = \dots, \mathbf{Y} = \dots, \mathbf{Z} = \dots$   
 $= * = 1, 91.4887 - 291.4887 - 220.1(87 -) - 17, 1.4887 - 1.4887 - 87 - 116 0 0 7.5716 177.3568 5 5256 6640 \dots$

#### 4. Comparison of $t_{\text{max}}$

As a first step in comparing the two models, we consider the time to reach maximum concentration,  $t_{\text{max}}$ , for the two models. For the one-dimensional model,  $t_{\text{max}}$  is given by

that  $C$  is not identically zero. Then given any  $a < b$ , there is a nonzero measure of initial states  $(x_0, x'_0)$  and a sequence  $c_t \in (V)_+ \cup (V)_-$  such that the solution of (14) has momenta,  $x_t = T_2(x_{t-1}, x'_t)$  satisfying  $a < x_t < b$  and  $x'_t > b$  for some time  $T$ .

**P** ... (14) ...  $c_- \in (V)_-, c_+ \in (V)_+, x_t = c_{\pm}, C(t) = \pm 1$ .

$$\tilde{L}(x, x') = T(x, x') + W(x) + \tilde{C}(x),$$

$$\tilde{C} = V(c_{\pm}(x)) \quad C(x) \geq 0.$$

$$\tilde{C}(x+2) - \tilde{C}(x) > 0, \quad \tilde{L}(x+2, x'+2) - \tilde{L}(x, x') \dots \mathbf{A} \dots \square$$

4.2. Standard example

$$L(x, x', t, t') = \frac{1}{2} (x' - x)^2 + \frac{1}{2} (t' - t)^2 + k \dots x(1 + h \dots), \tag{15}$$

$$k > 0, \quad h > 0, \quad \mathbf{A} \dots (1), \dots \tag{15}$$

$t \rightarrow \dots$ 
 $^* = 2 - m$ 
 $(0, 2 - m) \dots (2 - m)$ 
 $(16)$

$^* = (2m + 1)$ 
 $(0, ^*) \mapsto (, ^*)$ 
 $(16)$

$g$ 
 $g^*$ 
 $g^*$ 
 $g$ 
 $(16)$

$T^2 = \{(, ) : 0 < , < 2\}$ 
 $^* = (t_{+1} - t)^* = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^{T-1} (U'(t))$

$\langle ^* \rangle = \frac{1}{4 - 2} \int_{T^2} U'(\cdot) = \frac{1}{2}$

$(16)$



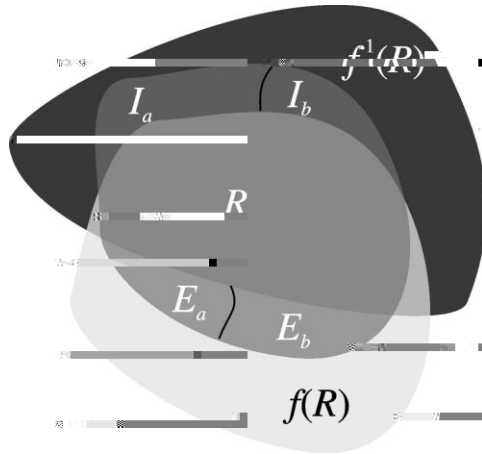


Fig. 4. The set  $R$  and its image  $f(R)$ .

where

$$E = \{z \in R : f(z) \notin R\} = R \setminus f^{-1}(R).$$

Since  $f$  is a contraction,  $\mu(f(R)) < \mu(R)$ . The set  $E$  is the part of  $R$  that does not map back into  $R$ .

$$\mu(E) = \mu(R \setminus f^{-1}(R)) = \mu(R) - \mu(R \cap f^{-1}(R)) = \mu(R) - \mu(f(R) \cap R) = \mu(R \setminus f(R)) = \mu(I). \quad (17)$$

Let  $S^t$  be the set of points in  $R$  that stay in  $R$  for  $t$  iterations of  $f$ .

$$S^0 = I, \quad S^t = f(S^{t-1}) \cap R = f(S^{t-1} \setminus E).$$

Since  $S^t \subset R$ ,  $\mu(S^t) < \mu(R)$ . The set  $S^t$  is the part of  $R$  that stays in  $R$  for  $t$  iterations of  $f$ . The set  $S^t$  is a subset of  $R$ , and  $\mu(S^t) < \mu(R)$ . The set  $S^t$  is the part of  $R$  that stays in  $R$  for  $t$  iterations of  $f$ . The set  $S^t$  is a subset of  $R$ , and  $\mu(S^t) < \mu(R)$ . The set  $S^t$  is the part of  $R$  that stays in  $R$  for  $t$  iterations of  $f$ .

$$\mu(p(I_a) \cap E_b) = \mu(p(I_a)) - \mu(p(I_a) \cap E_a) \geq \mu(I_a) - \mu(E_a). \quad \square$$

**A**  $f_t: R \rightarrow R$  is a sequence of measure-preserving homeomorphisms, and  $R$  is a measurable set with incoming sets  $I_t$  and exit sets  $E_t$ .

$$I_t = R \setminus f_t(R), \quad E_t = R \setminus f_t^{-1}(R).$$

(17)  $S_k^t = I_{k-1}$ ,  $S_k^{t+1} = f_t(S_k^t \setminus E_t)$ .

**A**  $\sum_{k=-\infty}^t \mu(S_k^t) < \mu(R)$

**L** **a 4.** Let  $f_t$  be a sequence of measure-preserving homeomorphisms, and  $R$  a measurable set with incoming sets  $I_t$  and exit sets  $E_t$ .

5.2. Maps of the cylinder

$$f(C) = \int_C (y', x' - y, x) = \int_C (y', x' - y, x)$$

Let  $C$  be a curve in  $T$  and  $B$  a subset of  $T$ . Then  $U \subset T$  is defined by

$$U = \{z \in T : f^{-1}(z) \in B\}.$$

Similarly,  $D \subset B$  is defined by

$$D = \{z \in B : f^{-1}(z) \in T\}.$$

$$\mu(U) - \mu(D) = \int_C (y', x' - y, x) = \int_C (y', x' - y, x)$$

**Proposition 5.** Suppose that  $f_t$  is a sequence of area and end-preserving homeomorphisms of the cylinder, and that the net flux  $\int_C (y', x' - y, x) \geq \epsilon > 0$ . Let  $A$  denote the annulus bounded by the circles  $\{y = a\}$  and  $\{y = b\}$  where  $a < b$ . Then, there is a set of positive measure of orbits that cross the annulus.

**Proof.** Let  $U_t(a)$  and  $D_t(a)$  be the sets defined by  $U_t(a) = \{z \in A : f_t^{-1}(z) \in \{y = a\}\}$  and  $D_t(a) = \{z \in A : f_t^{-1}(z) \in \{y = b\}\}$ . Similarly, let  $U_t(b)$  and  $D_t(b)$  be the sets defined by  $U_t(b) = \{z \in A : f_t^{-1}(z) \in \{y = b\}\}$  and  $D_t(b) = \{z \in A : f_t^{-1}(z) \in \{y = a\}\}$ . Then  $E_t = U_t(a) \cup D_t(b)$  and  $F_t = D_t(a) \cup U_t(b)$ . Since  $\mu(U_t(b)) \geq \epsilon > 0$ , there is a set of positive measure of orbits that cross the annulus.  $\square$

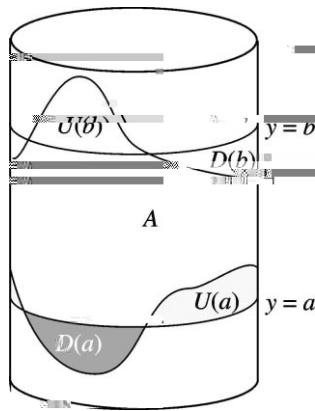


Fig. 5. A cylinder with an annulus A bounded by circles  $y = a$  and  $y = b$ . The sets  $U(a)$ ,  $D(a)$ ,  $U(b)$ , and  $D(b)$  are shown as regions within the cylinder.

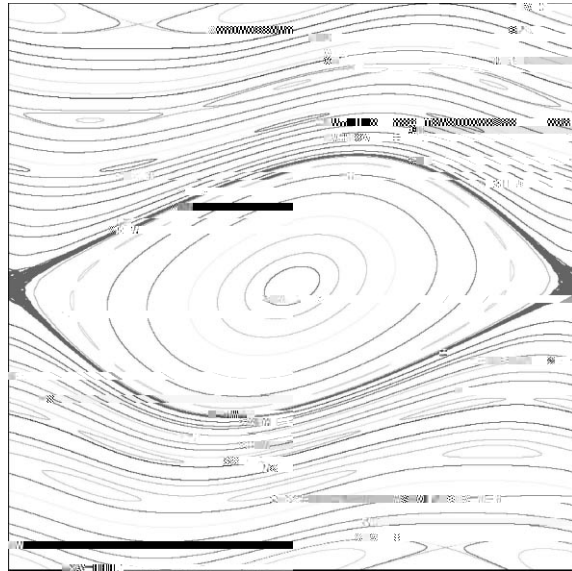


Fig. 6. Phase portrait of the standard map with net flux for  $k = 0.5$ . The ring of points is a limit cycle.

5.3. Standard map with net flux

Consider the standard map with net flux  $k$  on the cylinder  $(0, 2\pi) \times \mathbb{R}$ . The potential function is  $V(x) = V(x + 2\pi) = V(x) - V(0)$ . The map is given by

$$x' = x + y, \quad y' = y - k \sin(x) + \frac{1}{2} \sin(2x).$$

For  $k < k_{cr} \approx 0.971635406$ , the map has a unique fixed point at  $(0, 0)$ . For  $k = 0.5$ , the map has a unique fixed point at  $(0, 0)$  and a ring of points at  $y = 0$ . For  $k > k_{cr}$ , the map has a unique fixed point at  $(0, 0)$  and a ring of points at  $y = 2\pi m$ . The map is given by

$$f(x, y + 2\pi m) = f(x, y) + 2\pi(m, m).$$

The map is given by  $f(x, y) = (x + y, y - k \sin(x) + \frac{1}{2} \sin(2x))$ . The map is given by  $f(x, y) = (x + y, y - k \sin(x) + \frac{1}{2} \sin(2x))$ . The map is given by  $f(x, y) = (x + y, y - k \sin(x) + \frac{1}{2} \sin(2x))$ .

$$x = \frac{y}{2k}$$

The map is given by  $f(x, y) = (x + y, y - k \sin(x) + \frac{1}{2} \sin(2x))$ . The map is given by  $f(x, y) = (x + y, y - k \sin(x) + \frac{1}{2} \sin(2x))$ .

6. Periodic orbit

$$z_{t-1} = (x_{t-1}, x_t, t-1, t) \tag{12}$$

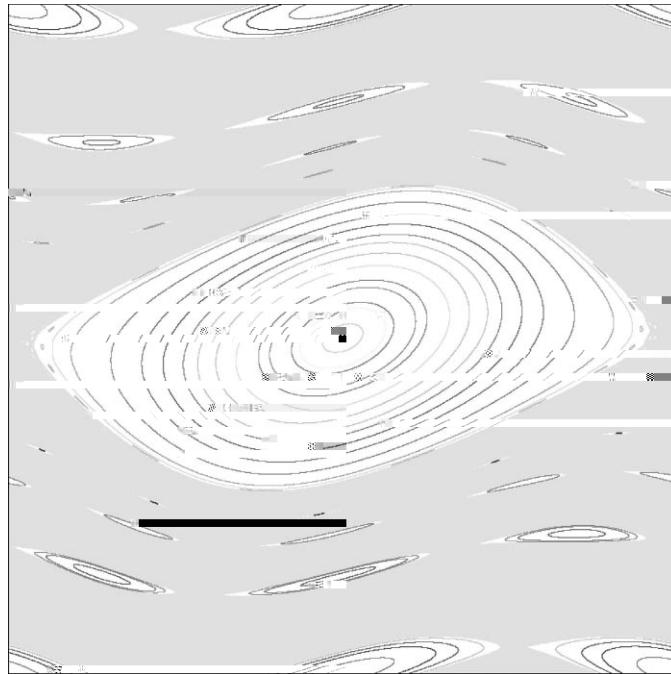


Fig. 7. Contour plot of the potential function  $V(x)$  for  $k = 0.5$ ,  $F = 4^{-2}/1000$ . The central region is bounded by a thick black bar.

$$z_t = (x_t, t) = \left( -x_{t-1} + 2x_t + \frac{1}{t} V(x_t) + C(t), -x_{t-1} + 2x_t + W(t) + V(x_t) - C(t) \right) \quad (19)$$

where  $C(t) = 0$ ,  $W(t) = 0$ ,  $x_t = c_t \in \mathcal{A} \cap (V)$ ,  $0 < C(t) < 1$ ,  $W(t) > 0$ ,  $x_t \in \mathcal{A}$ .

**Lemma 6.** Suppose that  $z_t$ , given by (19), is a  $C^2$  map of  $\mathbb{T}^4$ , such that  $1 + C(t) \geq \epsilon > 0$ . Then, for any sequence  $\{c_0, c_1, \dots\}$  with  $c_t \in \mathcal{A} \cap (V)$ , any initial condition  $(x_0, t_0)$ , and any  $\epsilon > 0$ , there exists an orbit  $z_t = (x_t, x_{t+1}, t, t+1)$ ,  $t \geq 0$  of  $z_t$  such that

$$|x_t - c_t| \leq \epsilon \quad t \geq 0,$$

provided

$$0 \leq \epsilon < \epsilon_0 = \frac{1}{(4 + a)}, \quad (20)$$

where  $(a, b) \equiv \sup_{t \geq 0} |V(c_t \pm a)|$ .



(1/2)  $V(c_{t+1}^+)$   $> (4 + a)$   $\dots$   $a$   $1 + C(\dots) \geq \dots$   $> 4 + a$  (20).  
 $\Delta$   $S$   $W_0$   $S = \mathbb{R}^2 \times (0, 1) \cap W_0$   $B_{t+1}$   $W_{t+1}$   
 $z_0$   $S$   $T$   $S$   
 $U_t$   $W_t$   $t \geq 1$   $B$   
 $W_0$   $T$   $B$   
 $T$   $B$   $W_t$   
 $S$   $(63.7 \dots 8089.31, 81, 17.0455$

For  $|t| \leq r^t$ ,  $r > 1$ ,  $r^2 - wr - 1 = 0$ ,  $w = \frac{1}{2}x(2 + |W(x)|)$ .  
 $|t| \leq \frac{1}{2}M^2 r^{2t}$ .

For  $T \leq t \leq T^2$ ,  $W = 0$ ,  $|t| \leq T^2$ .  $\square$

**R a**  $C(\cdot)$

6.1. Standard example, continued

(15),  $V(x) = k|x|$ ,  $C(\cdot) = h$ ,  $h < 1$ .  
 $a = 2$ ,  $6$ .  
 $\leq 0 = \frac{k(1-h)}{4+2}$ .

$M = kh$ ,  $W = 1$ ,  $DB)DB$ ,  $b B Db$ ,  $b b$ ,  $ET bDD B$



where  $h < 1$  and  $\leq 0$ . (9)

### 7. Conclusion

... 6,9,20 ...  
 ... 19 ...  
 ... 16 ...  
 ... (x, y) ...  
 ... 17 ...

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